Introduction to Functions of Several Variables
(Basic Definitions and Notation)

Section 1

Single Variable Calculus

Definition:

A real-valued function *f* of one variable is a rule that assigns to each real number *x* in a set D called the domain a unique real number *y* in a set R called the range.

We denote this by $y = f(x)$.

Single Variable Calculus

Domain of *f(x)*:

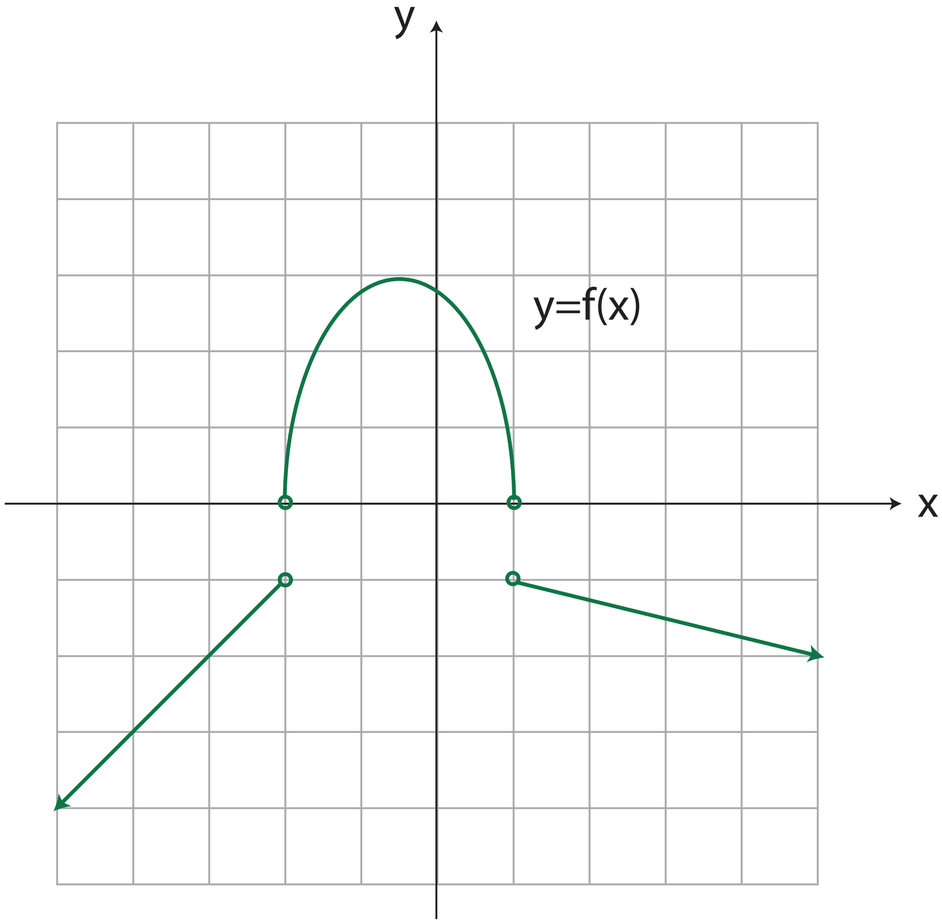
The set of all *x*-values for which *f(x)* is defined as a real number. (All possible *x*-values the equation will accept as input).

Range of *f(x)*:

The set of all *y*-values that *f* can attain.
(All possible output values).

Single Variable Calculus

The **graph** of a function *f* is a set of all ordered pairs (points) (x,y) where x is in the domain of *f* and $y=f(x)$.



Functions of Two Variables

Definition:

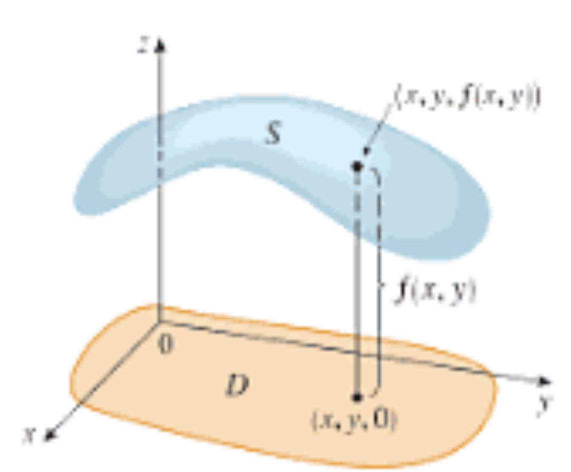
A real-valued function *f* of two variables is a rule that assigns to each ordered pair of real numbers (x,y) in a set *D* called the domain a unique real number *z* in a set *R* called the range.

We denote this by

$$z = f(x,y).$$

Functions of Two Variables

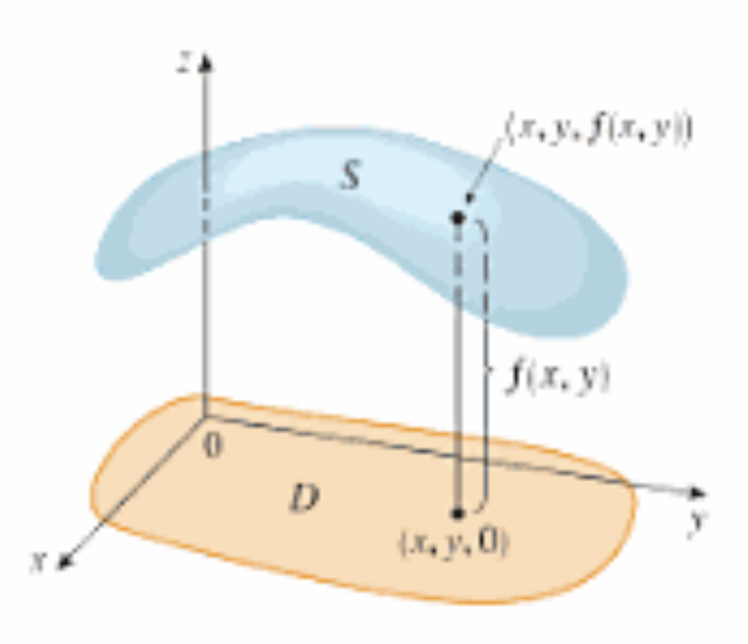
Domain of *f(x,y)*:
The set of all ordered pairs (x,y) for which f(x,y) is a real number. (A subset of the xy-plane, R^2).



Range of *f*(*x*,*y*):
The set of all z-values that f can attain. (A subset of the real number line, R).

Functions of Two Variables

The **graph** of a function $z=f(x,y)$ of two variables is the set of points (x,y,z) in the space R^3 such that $z=f(x,y)$ for some (x,y) in the domain of f.



Functions of Two Variables

Example: Body Mass Index

$$BMI(m,h) = \frac{m}{h^2}$$

where m is a person's mass in kilograms
and h their height in metres.

BMI is the **dependent variable;**
m and h are the two **independent variables**.

Functions of Two Variables

Example: <u>Body Mass Index</u>

WEIGHT lbs	100	105	110	115	120	125	130	135	140	145	150	155	160	165	170	175	180	185	190	195	200	205	210	215
kgs	45.5	47.7	50.0	52.3	54.5	56.8	59.1	61.4	63.6	65.9	68.2	70.5	72.7	75.0	77.3	79.5	81.8	84.1	86.4	88.6	90.9	93.2	95.5	97.7
HEIGHT in/cm	Underweight					Healthy					Overweight					Obese				Extremely obese				
5'0" - 152.4	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
5'1" - 154.9	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	36	37	38	39	40
5'2" - 157.4	18	19	20	21	22	22	23	24	25	26	27	28	29	30	31	32	33	33	34	35	36	37	38	39
5'3" - 160.0	17	18	19	20	21	22	23	24	24	25	26	27	28	29	30	31	32	32	33	34	35	36	37	38
5'4" - 162.5	17	18	18	19	20	21	22	23	24	24	25	26	27	28	29	30	31	31	32	33	34	35	36	37
5'5" - 165.1	16	17	18	19	20	20	21	22	23	24	25	25	26	27	28	29	30	30	31	32	33	34	35	35
5'6" - 167.6	16	17	17	18	19	20	21	21	22	23	24	25	25	26	27	28	29	29	30	31	32	33	34	34
5'7" - 170.1	15	16	17	18	18	19	20	21	22	22	23	24	25	25	26	27	28	29	29	30	31	32	33	33
5'8" - 172.7	15	16	16	17	18	19	19	20	21	22	22	23	24	25	25	26	27	28	28	29	30	31	32	32
5'9" - 175.2	14	15	16	17	17	18	19	20	20	21	22	22	23	24	25	25	26	27	28	28	29	30	31	31
5'10" - 177.8	14	15	15	16	17	18	18	19	20	20	21	22	23	23	24	25	25	26	27	28	28	29	30	30
5'11" - 180.3	14	14	15	16	16	17	18	18	19	20	21	21	22	23	23	24	25	25	26	27	28	28	29	30
6'0" - 182.8	13	14	14	15	16	17	17	18	19	19	20	21	21	22	23	23	24	25	25	26	27	27	28	29
6'1" - 185.4	13	13	14	15	15	16	17	17	18	19	19	20	21	21	22	23	23	24	25	25	26	27	27	28
6'2" - 187.9	12	13	14	14	15	16	16	17	18	18	19	19	20	21	21	22	23	23	24	25	25	26	27	27
6'3" - 190.5	12	13	13	14	15	15	16	16	17	18	18	19	20	20	21	21	22	23	23	24	25	25	26	26
6'4" - 193.0	12	12	13	14	14	15	15	16	17	17	18	18	19	20	20	21	22	22	23	23	24	25	25	26

BMI Chart

Functions of Two Variables

Example: Body Mass Index

Compute $BMI(60,h)$ and $BMI(m,1.7)$ and analyze the resulting functions.

What is the natural domain of BMI? What is its restricted domain?

Domain

Example:

Find and sketch the domain of each function.

(a) $f(x,y) = \ln(x + y - 1)$

(b) $h(x,y) = \dfrac{3xy}{x - xy^2}$

Range

Example:

Determine the range of each function.

(a) $f(x,y)=\ln(x+y-1)$ (b) $g(x,y)=e^{1-x^2-y^2}$

Functions of Two Variables

Linear Functions:
Linear functions in two variables are of the form

$$f(x,y) = ax + by + c$$

'linear' because the exponent of both *x* and *y* is 1

where *a*, *b*, and *c* are real numbers.

<u>Domain</u>: all of R^2

<u>Graph</u>: plane

<u>Example</u>: $f(x,y) = 6 - 3x - 2y$

*Note: A linear functions is just a special case of a polynomial function (next)

Functions of Two Variables

Polynomial Functions:
A polynomial functions in two variables is a sum of terms of the form

$$cx^k y^l$$

where *c* is a real number and *k* and *l* are non-negative integers.

Domain: all of R^2

Examples:

$$f(x,y) = 1 - x^2 - y^2 \qquad g(x,y) = 3xy + x^4 y^3 - 1$$

Functions of Two Variables

Rational Functions:
A rational function in two variables is a quotient of two polynomials in two variables.

Domain: all of R^2 except points at which the denominator = 0

Examples:

$$f(x,y) = \frac{x-y}{1+x^2+y^2} \qquad g(x,y) = \frac{3xy+x^4y^3-1}{x^2-y^2}$$

Graphs

Example:

Sketch the graphs of each function.

(a) $f(x,y) = \sqrt{x^2 + y^2}$

(b) $g(x,y) = 1 - x^2 - y^2$

(c) $h(x,y) = \sqrt{1 - x^2 - y^2}$