Limits and Continuity

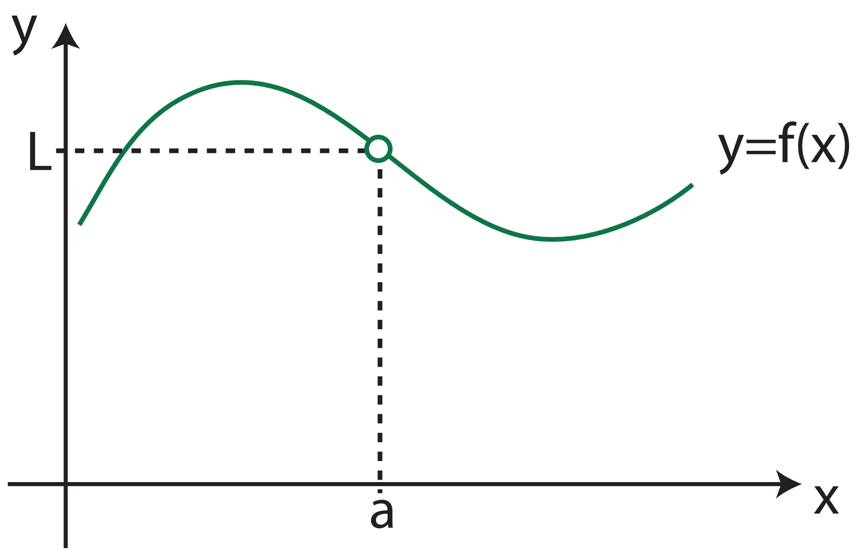
Section 3

Limit of a Function in R^2

Definition:

$$\lim_{x \to a} f(x) = L$$

means that the *y*-values can be made arbitrarily close (as close as we'd like) to *L* by taking the *x*-values sufficiently close to *a*, from either side of *a*, but not equal to *a*.

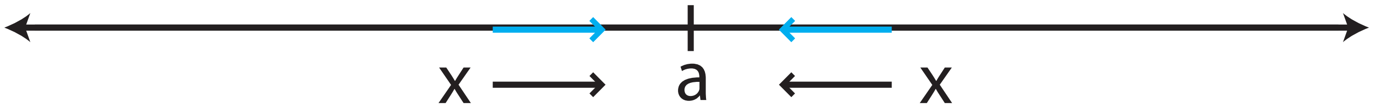


Existence of a Limit in R^2

The limit exists if and only if the left and right limits both exist (equal a real number) and are the same value.

Existence of a Limit in R^2

It is relatively easy to show that this type of limit exists since there are only <u>two</u> ways to approach the number a along the real number line: *either from the left or from the right*

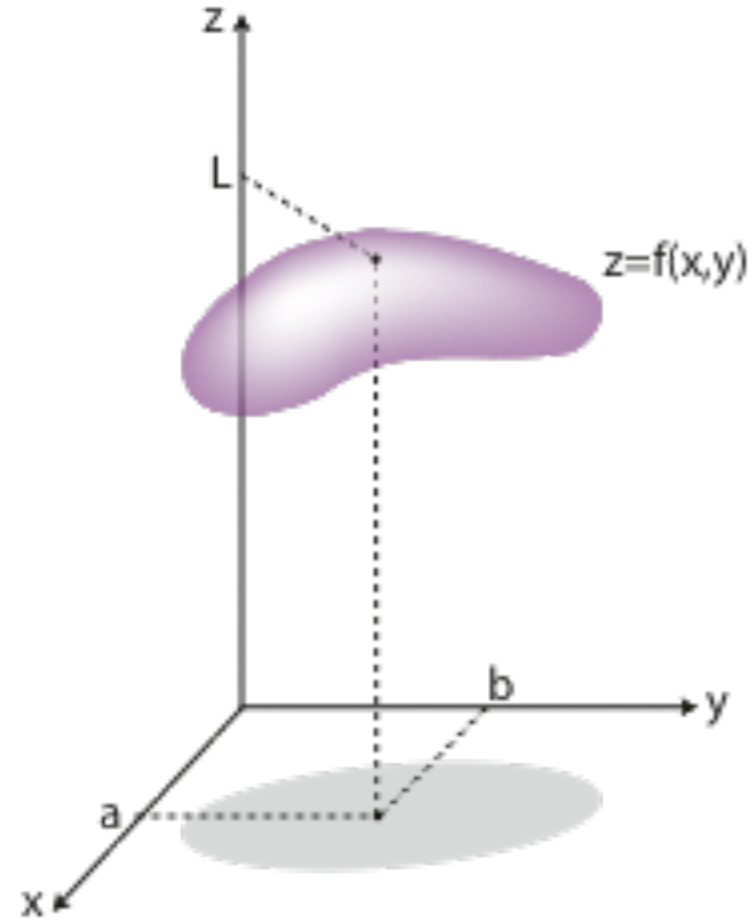


Limit of a Function in R^3

<u>Definition</u>:

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

means that the *z*-values approach *L* as (*x*,*y*) approaches (*a*,*b*) along <u>*every path*</u> in the domain of *f*.



Existence of a Limit in R^3

In general, it is difficult to show that such a limit exists because we have to consider the limit along all possible paths to (*a*,*b*).

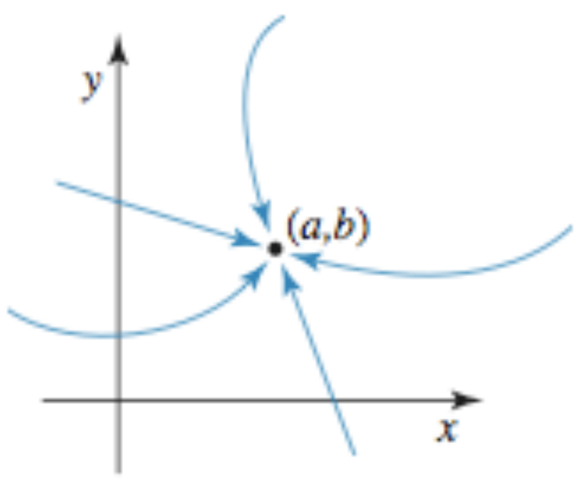


FIGURE 3.2 Paths leading to (a, b)

Existence of a Limit in R^3

However, to show that a limit **doesn't** exist, all we have to do is to find two *different* paths leading to (*a*,*b*) such that the limit of the function along each path is different (or does not exist).

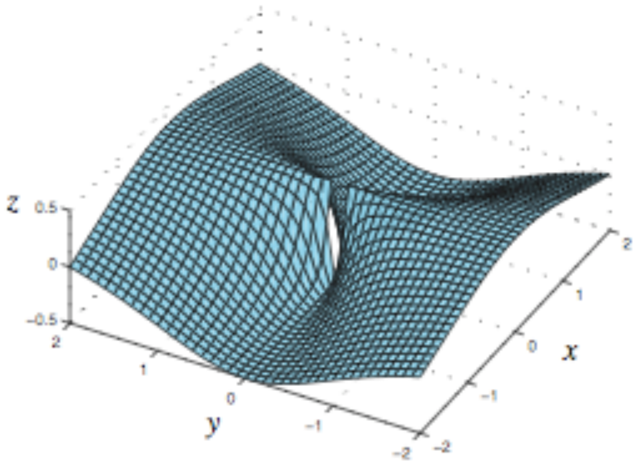


FIGURE 3.4 The graph of $f(x, y) = \frac{y^2 - x^2}{2x^2 + 3y^2}$

Existence of a Limit in R^3

Example:

Show that the following limits **do not** exist.

(a) $$\lim_{(x,y)\to(0,0)} \frac{y^2 - x^2}{2x^2 + 3y^2}$$

(b) $$\lim_{(x,y)\to(0,0)} \frac{6x^3 y}{2x^4 + y^4}$$

(c) $$\lim_{(x,y)\to(0,0)} \frac{x^2 + \sin^2 y}{2x^2 + y^2}$$

Limit Laws

Theorem:
Assume that $\lim_{(x,y)\to(a,b)} f(x,y)$ and $\lim_{(x,y)\to(a,b)} g(x,y)$ exist (i.e. are real numbers). Then

(a) $\lim_{(x,y)\to(a,b)} \left(f(x,y) \pm g(x,y)\right) = \lim_{(x,y)\to(a,b)} f(x,y) \pm \lim_{(x,y)\to(a,b)} g(x,y)$

(b) $\lim_{(x,y)\to(a,b)} \left(c\,f(x,y)\right) = c \lim_{(x,y)\to(a,b)} f(x,y)$, where c is any constant.

Limit Laws

Theorem (continued):

(c) $$\lim_{(x,y)\to(a,b)}\big(f(x,y)\times g(x,y)\big)=\lim_{(x,y)\to(a,b)}f(x,y)\times\lim_{(x,y)\to(a,b)}g(x,y)$$

(d) $$\lim_{(x,y)\to(a,b)}\frac{f(x,y)}{g(x,y)}=\frac{\lim\limits_{(x,y)\to(a,b)}f(x,y)}{\lim\limits_{(x,y)\to(a,b)}g(x,y)},\quad \text{provided}\lim_{(x,y)\to(a,b)}g(x,y)\neq 0.$$

Some Basic Rules

For the function $f(x,y)=x$, $\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} x = a$

For the function $f(x,y)=y$, $\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} y = b$

For the function $f(x,y)=c$, $\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} c = c$

Evaluating Limits

Example #10:
Using the properties of limits, evaluate

$$\lim_{(x,y)\to(2,-2)} \frac{1}{xy-4}.$$

Solution:

$$\lim_{(x,y)\to(2,-2)} \frac{1}{xy-4}$$

$$= \frac{\lim_{(x,y)\to(2,-2)} 1}{\lim_{(x,y)\to(2,-2)} (xy-4)}$$

$$= \frac{\lim_{(x,y)\to(2,-2)} 1}{\lim_{(x,y)\to(2,-2)} x \cdot \lim_{(x,y)\to(2,-2)} y - \lim_{(x,y)\to(2,-2)} 4}$$

$$= \frac{1}{2\cdot(-2)-4}$$

$$= -\frac{1}{8}$$

Direct Substitution

Theorem:

If $f(x,y)$ is a polynomial or rational function (in which case (a,b) must be in the domain of f), then

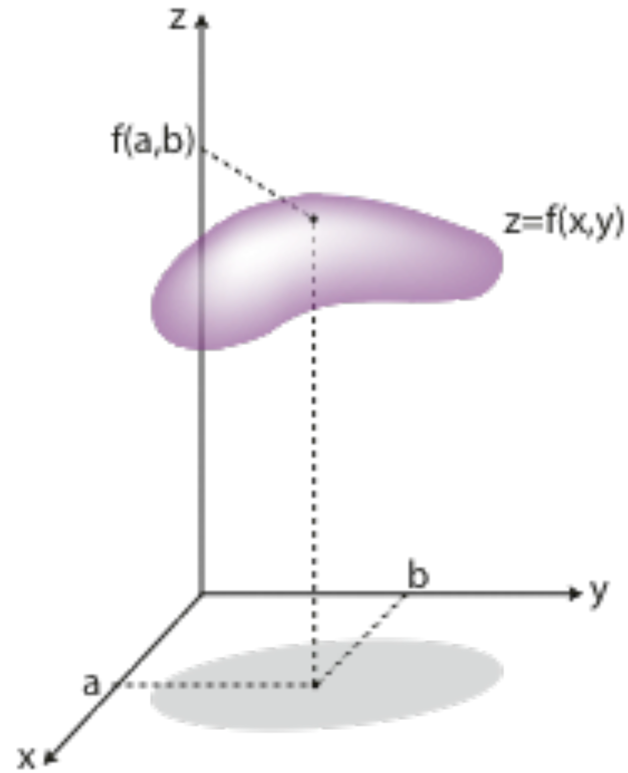
$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Continuity of a Function in R^3

Intuitive idea:

A function is continuous if its graph has no holes, gaps, jumps, or tears.

A continuous function has the property that a small change in the input produces a small change in the output.

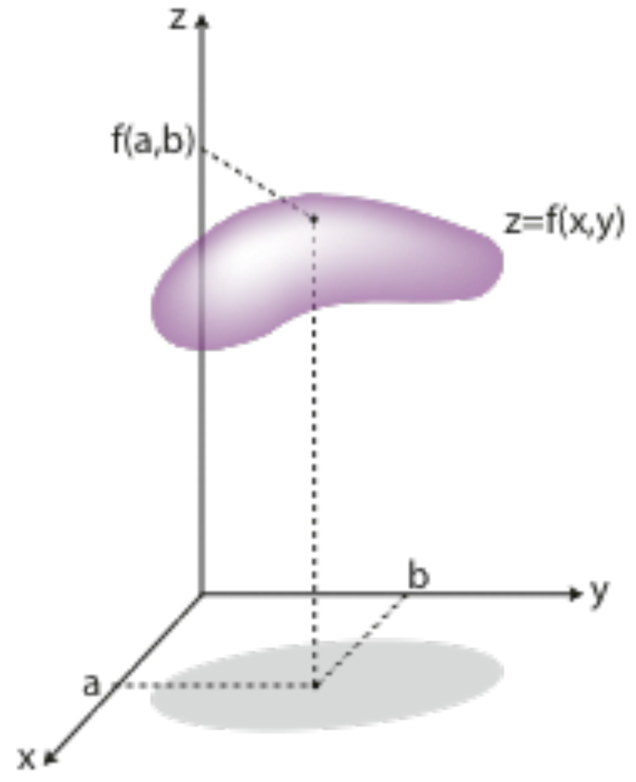


Continuity of a Function in R^3

Definition:

A function f is continuous at the point (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$



Continuity of a Function in R^3

Example:

Determine whether or not the function

$$f(x,y)=\begin{cases} x^2+y^2+4 & \text{if } (x,y)\neq(0,0) \\ 1 & \text{if } (x,y)=(0,0) \end{cases}$$

is continuous at (0,0).

Which Functions Are Continuous?

A function is **continuous** if it is continuous at every point in its domain.

Basic Continuous Functions:

- ✓ polynomials
- ✓ rational functions
- ✓ exponential functions
- ✓ logarithmic functions
- ✓ trigonometric functions
- ✓ root functions

Which Functions Are Continuous?

Combining Continuous Functions:
The sum, difference, product, quotient, and composition of continuous functions is continuous where defined.

Example:
Find the largest domain on which $f(x,y) = e^{x^2 y} + \sqrt{x + y^2}$ is continuous.

Limits of Continuous Functions

By the definition of continuity, if a function is continuous at a point, then we can evaluate the limit simply by **direct substitution**.

Example: Evaluate $\lim_{(x,y)\to(0,-1)}\left(e^{x^2 y}+\sqrt{x+y^2}\right)$