Tangent Plane, Linearization, and Differentiability

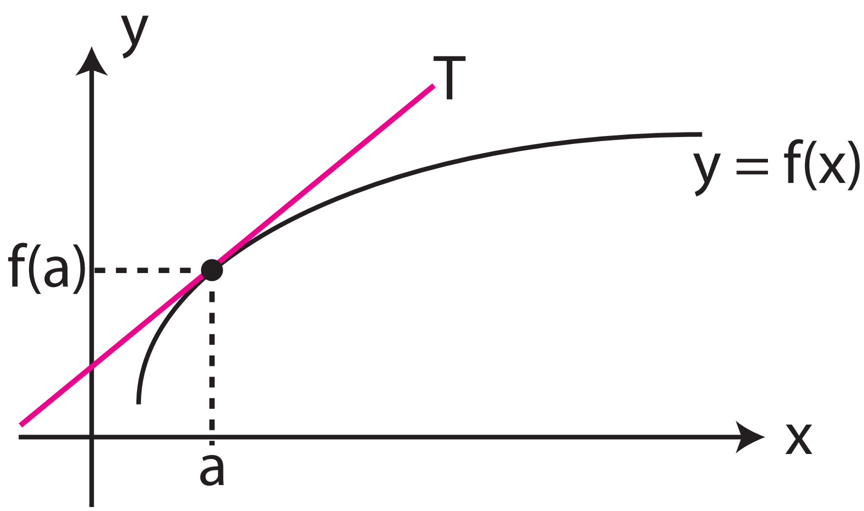
Section 5

Tangent Lines

Let $y=f(x)$ be a differentiable function in R^2.

Equation of the tangent line to the graph of *f* at $(a,f(a))$:

$$y - f(a) = f'(a)(x - a)$$



Linearization of *f* at $x=a$:

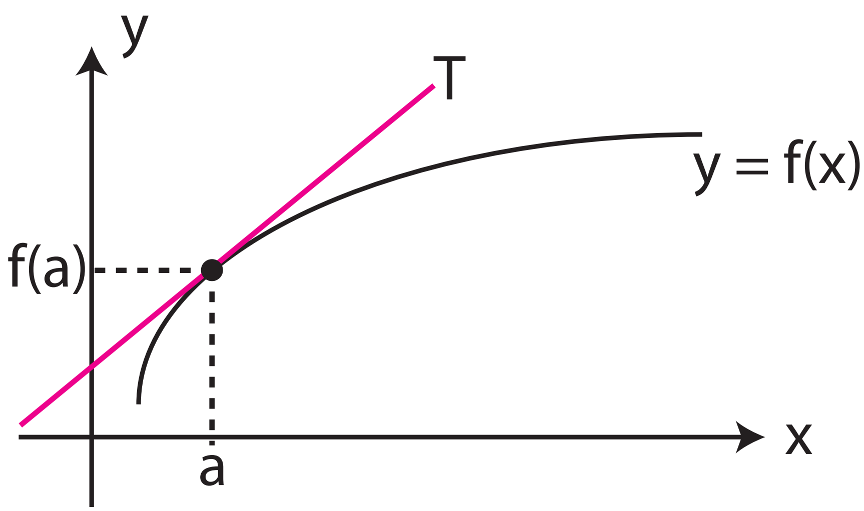
$$L_a(x) = f(a) + f'(a)(x - a)$$

L because this is a Linear function

Tangent Lines

The function *f* is approximately equal to its linearization at (a, $f(a)$) when the value of x is close to a.

as you zoom in around (a, $f(a)$), the line T more and more closely resembles the curve *f*

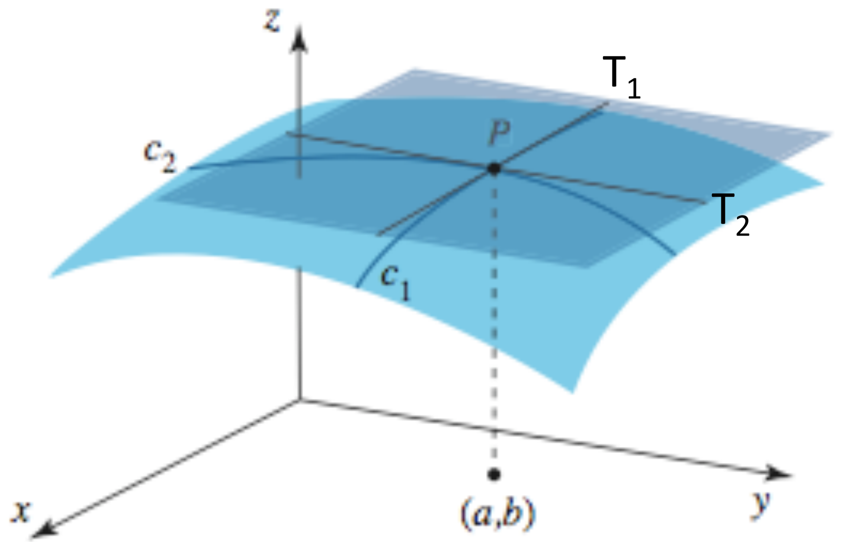


Linear approximation of *f* at *x*=*a*:

$$f(x) \approx f(a) + f'(a)(x-a)$$

Tangent Planes

Let $z=f(x,y)$ be a function in R^3 with continuous partial derivatives f_x and f_y.



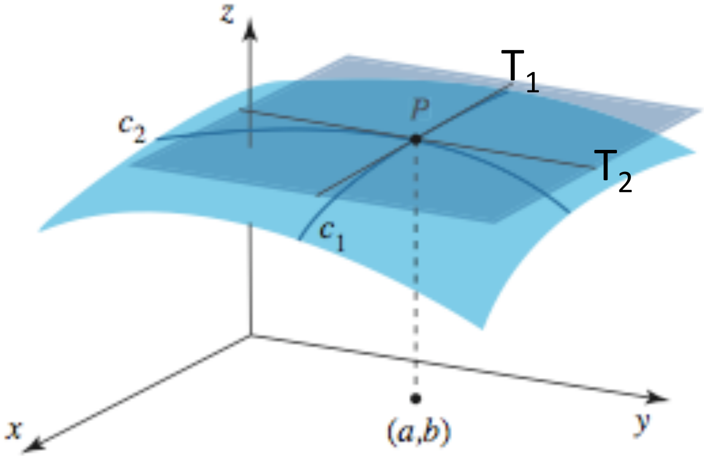
<u>Definition</u>: **Tangent Plane**

The plane that contains the point P and the tangent lines T_1 and T_2 at P is called the *tangent plane to the surface z=f(x,y) at P.*

Tangent Planes

Equation of the tangent plane to the surface $z=f(x,y)$ at $(a, b, f(a,b))$:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$



as you zoom in around $(a,b, f(a,b))$, the tangent plane more and more closely resembles the surface f

Tangent Planes

Example:

Find an equation of the tangent plane to the surface $f(x, y) = \ln(x - 3y)$ at the point (7, 2).

Linearization and Linear Approximation

<u>Definition</u>:
Assume that $z=f(x,y)$ has continuous partial derivatives at (a,b).

Linearization of *f* at (*a*,*b*):

$$L_{(a,b)}(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Linear approximation (or tangent plane approximation) of *f* at (*a*,*b*):

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Linearization and Linear Approximation

Example:

Find the linearization of $f(x,y)=\ln(x-3y)$ at $(7,2)$ and use it to approximate $f(6.9,\,2.06)$.

Differentiability in R^2

- A function $f(x)$ is differentiable at a point $x=a$ if $f'(a)$ exists,

i.e. if the limit

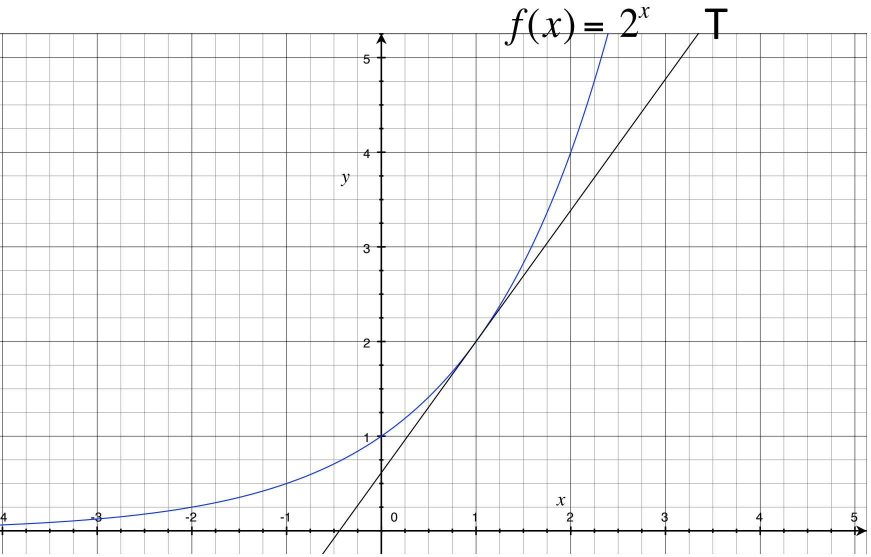
$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

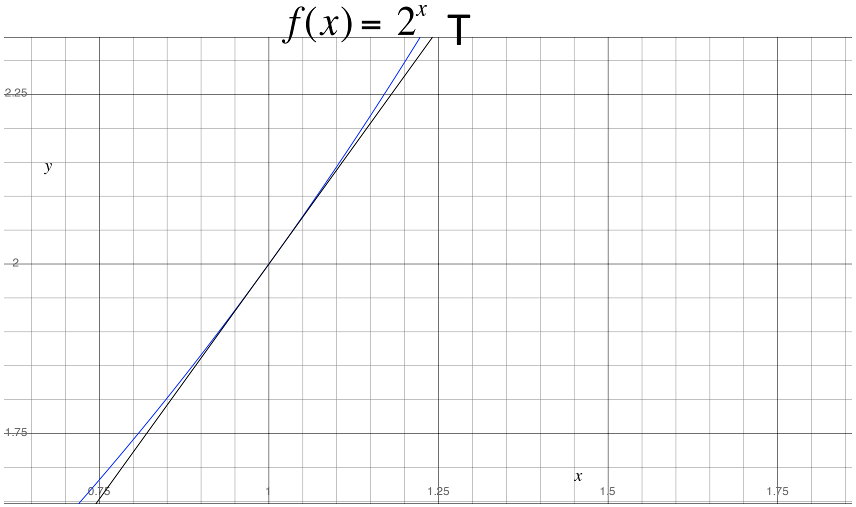
equals a real number.

Differentiability in R^2

- Geometrically, a function is differentiable at $x=a$ if its tangent line is *well-defined* at $(a,f(a))$.
- A well-defined tangent line has the property that it closely resembles the graph of the function on both sides of $x=a$ as we move closer and closer to the point (i.e., as we zoom in around the point, the curve and its tangent line become indistinguishable).

Differentiability in R^2

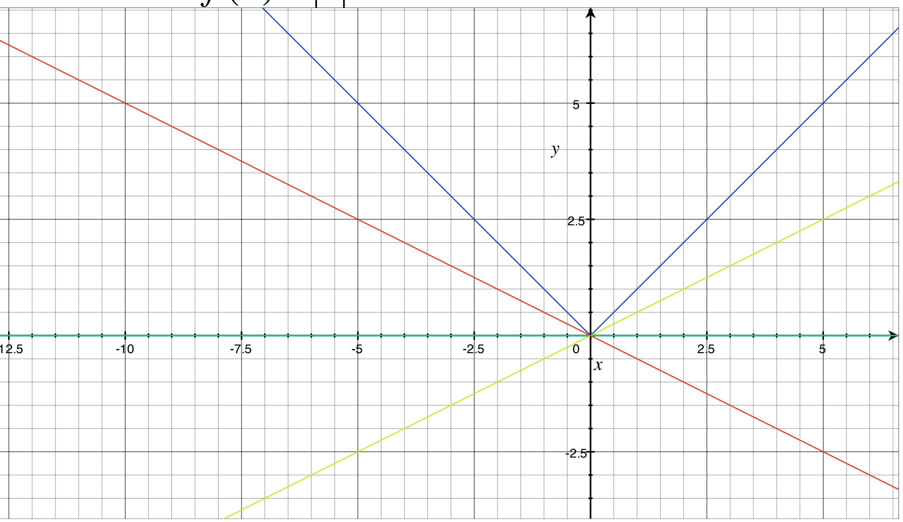




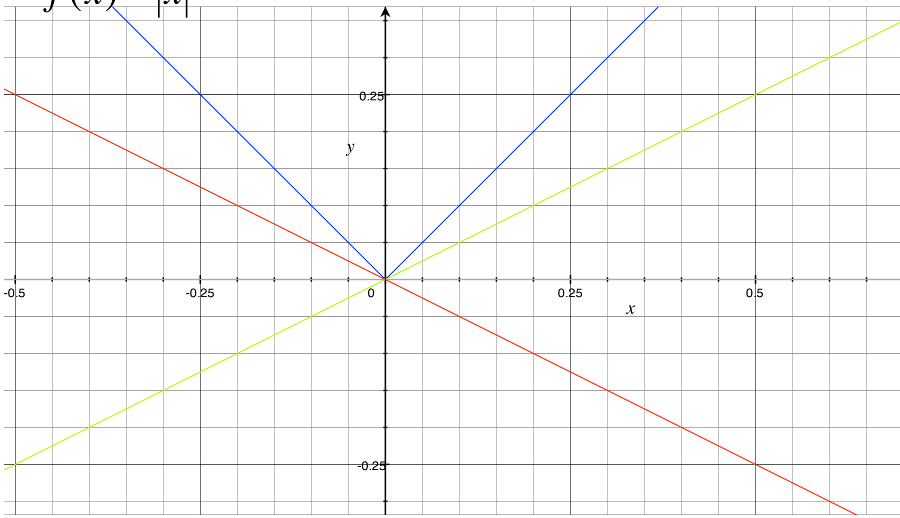
* $f(x)$ is differentiable at x=1

Differentiability in R^2

$f(x) = |x|$



$f(x) = |x|$



* $f(x)$ is NOT differentiable at x=0

Differentiability in R^3

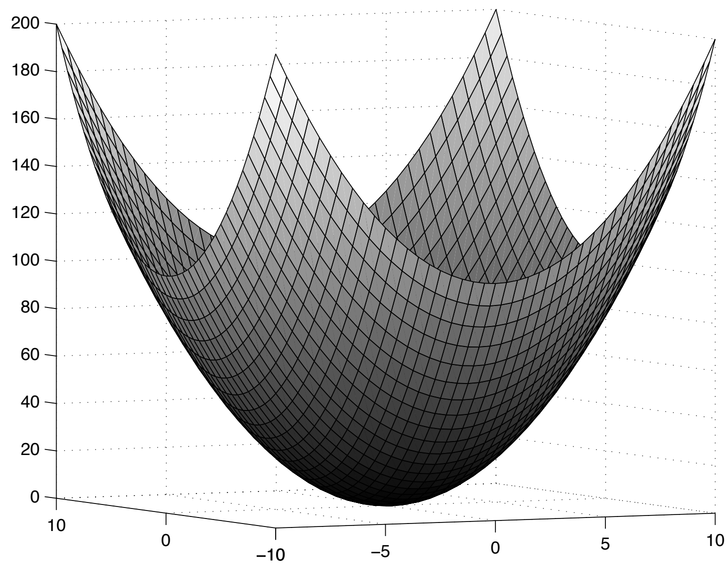
- Theoretically, a function $f(x,y)$ is differentiable at $(x,y)=(a,b)$ if the *directional* derivative of f exists in *EVERY* direction at (a,b). (This is impossible to check directly using the algebraic definition of the derivative.)

Differentiability in R^3

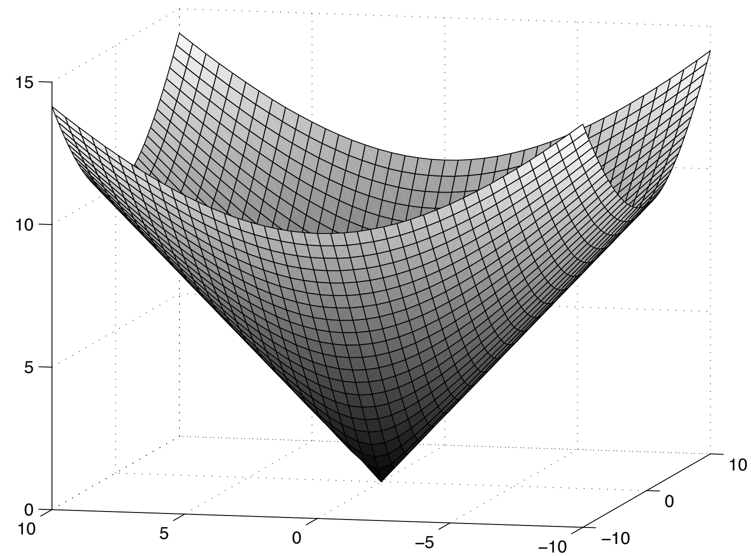
- Geometrically, a function $f(x,y)$ is differentiable at a point $(x,y)=(a,b)$ if its tangent **plane** is *well-defined* at (a,b).
- A well-defined tangent plane has the property that it closely resembles the graph of the function ***all around*** the point (a,b) as we move closer and closer to the point (i.e., as we zoom in around the point, the surface and its tangent plane become indistinguishable).

Differentiability in R^3

$$f(x) = x^2 + y^2$$



$$f(x) = \sqrt{x^2 + y^2}$$



Differentiable at (0,0)

NOT differentiable at (0,0)

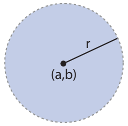
Differentiability for a Function of Two Variables

When a function $f(x,y)$ is differentiable at a point (a,b), we say that its linearization $L_{(a,b)}(x,y)$ is a good approximation to f near (a,b) and so the linear approximation

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

is valid for (x,y) near (a,b).

Theorems



Sufficient Condition for Differentiability
Assume that f is defined on an open disk $B_r(a,b)$ centred at (a,b), and that the partial derivatives f_x and f_y are continuous on $B_r(a,b)$. Then f is differentiable at (a,b).

Differentiability Implies Continuity
Assume that a function f is differentiable at (a,b). Then it is continuous at (a,b).

Differentiability for a Function of Two Variables

Example:

Verify that the linear approximation

$$\frac{2x+3}{4y+1} \approx 3 + 2x - 12y$$

is valid for (x,y) near (0,0).

Differentiability for a Function of Two Variables

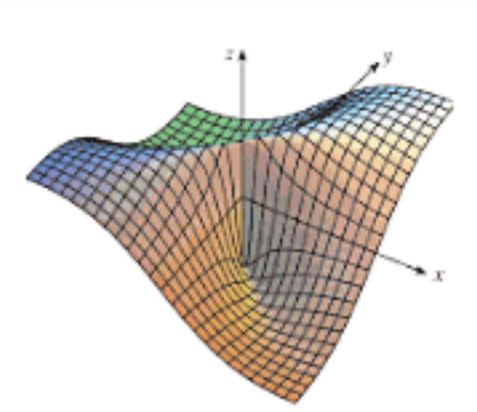
Example #16.

Show that the function $f(x,y) = x\tan y$ is differentiable at (0,0). What is the largest open disk centred at (0,0) on which *f* is differentiable?

Differentiability for a Function of Two Variables

Example in your text:

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & if \quad (x,y) \neq (0,0) \\ 0 & if \quad (x,y) = (0,0) \end{cases}$$



Using the formula, and ignoring the fact that the partial derivatives are ***<u>not</u>*** continuous at (0,0), we find the linearization (tangent plane approximation) to be

$$L_{(0,0)}(x,y) = 0$$

Differentiability for a Function of Two Variables

Example in your text:

However this is <u>*not*</u> a good approximation since the error between this linearization and the function does not approach 0 as (*x*,*y*) approaches (0,0).

For instance, along *y*=*x*, *f*(*x*,*x*)= ½ and the difference between the tangent plane and and the surface will remain constant at ½ (i.e. will not go to zero):

$$\text{error} = \left| f(x,y) - L_{(0,0)}(x,y) \right| = \left| \frac{1}{2} - 0 \right| = \frac{1}{2}$$