# Second-Order Partial Derivatives

Section 7

# Second-Order Partial Derivatives

Let *f* be a differentiable function of two variables, *x* and *y*.
Then *f* has two partial derivatives:

$$\frac{\partial f}{\partial x} \qquad \frac{\partial f}{\partial y}$$

Differentiating these expressions again (i.e. finding partial derivatives of partial derivatives), we obtain four second-order partial derivatives.

# Second-Order Partial Derivatives

Differentiating $\frac{\partial f}{\partial x}$ with respect to *x*, we obtain:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \quad \text{or} \quad (f_x)_x = f_{xx}$$

Differentiating $\frac{\partial f}{\partial y}$ with respect to *y*, we obtain:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} \quad \text{or} \quad (f_y)_y = f_{yy}$$

# Second-Order Partial Derivatives

Differentiating $\frac{\partial f}{\partial x}$ with respect to *y*, we obtain:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \quad \text{or} \quad (f_x)_y = f_{xy}$$

Differentiating $\frac{\partial f}{\partial y}$ with respect to x, we obtain:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} \quad \text{or} \quad (f_y)_x = f_{yx}$$

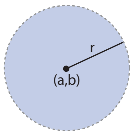
# Second-Order Partial Derivatives

**Example:**

Compute all second-order derivatives for the function $f(x, y) = \ln(2x + 3y)$.

# Equality of Mixed Partial Derivatives

**Theorem:**
Assume that a function $f$ is defined in an open disk $B_r(a,b)$ and that the partial derivatives $f_{xy}$ and $f_{yx}$ are continuous on $B_r(a,b)$ .
Then $f_{xy}(a,b)=f_{yx}(a,b)$.



# Second-Order Partial Derivatives

**Some ways in which these derivatives are useful:**

1. Second-order derivatives can help us to determine the behaviour of first-order derivatives and the concavity of the graph of f(x,y)

2. We can use second-order partial derivatives build partial differential equations which are used to model many real-life phenomena

3. The ‘Second Derivatives Test’ helps classify critical points of a function f(x,y) as local maxima, local minima, or saddle points

# Second-Order Partial Derivatives

**Example:**
Using the table of values below, determine whether the following partial derivatives are positive, negative, or zero:

$$f_x(4,1), \quad f_{xx}(4,1), \quad f_y(4,1), \quad f_{yy}(4,1)$$

| | x=3 | x=4 | x=5 | x=6 |
|---|---|---|---|---|
| y=0 | 5.9 | 6.1 | 6.8 | 6.7 |
| y=1 | 5.6 | 6 | 6.2 | 6.3 |
| y=2 | 5.4 | 5.7 | 6.1 | 6.5 |