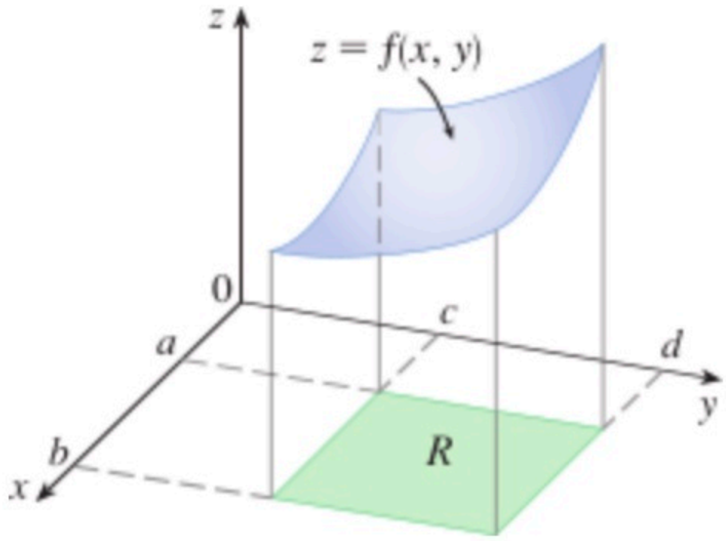
Double Integrals over Rectangles

Section 15.1

Volumes and Double Integrals

Consider a function $f(x, y) \geq 0$ defined on a closed rectangle

$$R = [a,b] \times [c,d] = \left\{ (x,y) \in R^2 \middle| a \leq x \leq b, c \leq y \leq d \right\}$$



Volumes and Double Integrals

Let *S* be the solid that lies above *R* and under the graph of *f*, i.e.,

$$S = \left\{ (x, y, z) \in R^3 \,\middle|\, 0 \le z \le f(x, y), (x, y) \in R \right\}$$

What is the volume of the solid *S*?

Estimating Volume

Divide *R* into subrectangles in the following way:

Divide [*a*,*b*] into *m* subintervals of width $\Delta x = \dfrac{b-a}{m}$.

Divide [*c*,*d*] into *n* subintervals of width $\Delta y = \dfrac{d-c}{n}$.

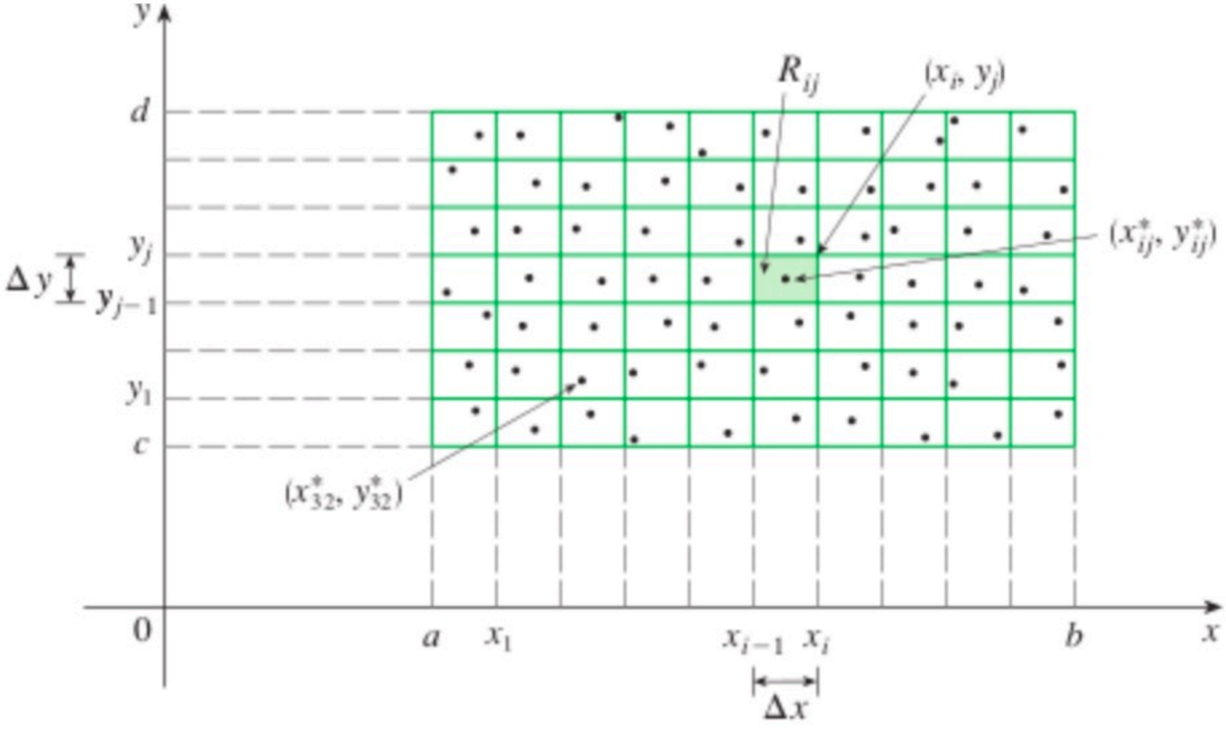
Note that we have *mn* subrectangles of the form

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$$

where the area of each subrectangle is $\Delta A = \Delta x \Delta y$.

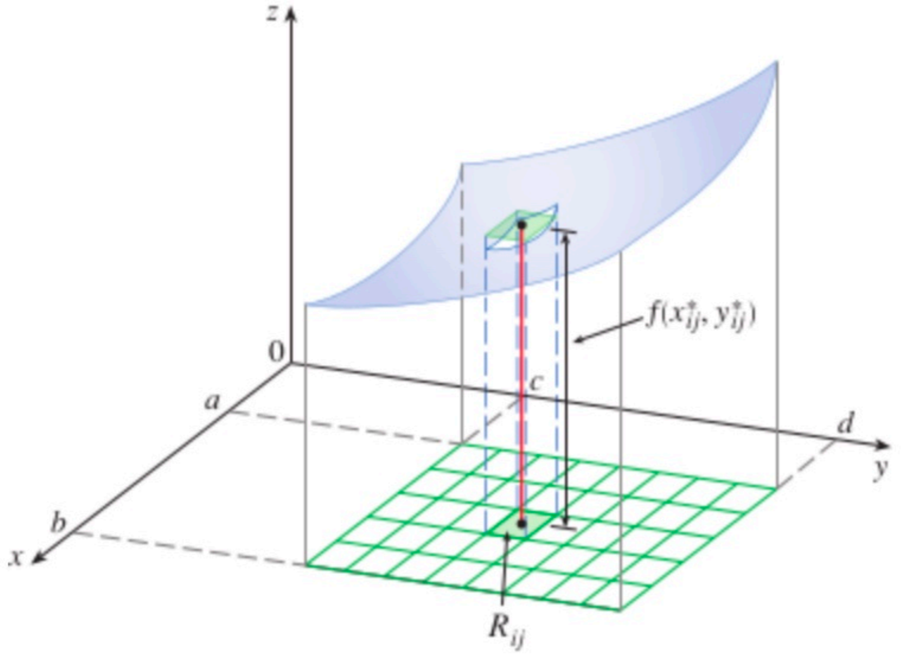
Estimating Volume

Next, choose a sample point (x_{ij}^*, y_{ij}^*) in each subrectangle and compute $f(x_{ij}^*, y_{ij}^*)$.



Estimating Volume

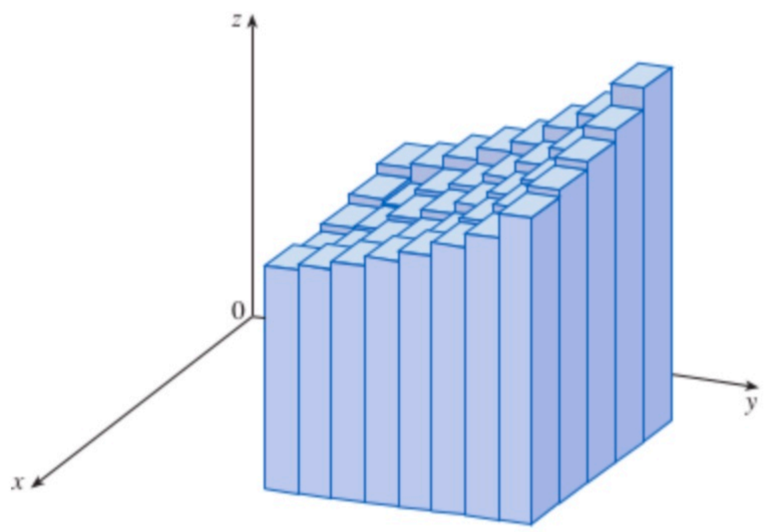
We can approximate the volume under the surface *f(x,y)* and above subrectangle R_{ij} by the rectangular prism with volume $f(x_{ij}^*, y_{ij}^*)\Delta A$.



Estimating Volume

Repeating this for all subrectangles, and adding together the results, we estimate the volume of the solid by the double Riemann sum:

$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*})\Delta A$$



Note: This approximation improves as *m* and *n* become larger.

The Double Integral

The double integral of *f* over the rectangle *R* is

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*})\,\Delta A$$

if this limit exists.

Note: When this limit exists, the function *f* is said to be **integrable**.

The Double Integral

When $f(x,y) \geq 0$, the double integral can be interpreted as the volume of the solid *S* that lies under the surface *f*(*x,y*) and above the rectangle *R*, i.e.,

$$V = \iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

The Midpoint Rule

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x}_i,\overline{y}_j)\Delta A$$

where $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

Exercise

Example, #4 modified.
Estimate the volume of the solid that lies below the surface $z = 1 + x^2 + 3y$ and above the rectangle $R = [1,2] \times [0,3]$

(a) using a Riemann sum with $m = n = 2$ and choosing sample points to be upper right corners.

***** ** Please work through the next example on your own. Solutions will be posted in the Solutions section of our OneNote Notebook *****

(b) Using the Midpoint Rule.

Partial Integration

Consider a function *f(x,y)* that is integrable on the rectangle $R = [a,b] \times [c,d]$.

Partial integration with respect to x:

$$\int_a^b f(x,y)\,dx$$

Partial integration with respect to y:

$$\int_c^d f(x,y)\,dy$$

Iterated Integrals

Suppose that a function *f*(*x,y*) is integrable on the rectangle $R = [a,b] \times [c,d]$. Then

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$$

and

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

Note: Always work from the inside out!

Iterated Integrals

Calculate each iterated integral.

(a) $\int_0^1 \int_1^3 (6x^2 y - 4x)\,dy\,dx$

***** Please work through the next example on your own. Solutions will be posted in the Solutions section of our OneNote Notebook *****

(b) $\int_0^1 \int_0^2 y e^{x-y}\,dx\,dy$

Fubini's Theorem

If f is continuous on the rectangle

$$R = \left\{ (x,y) \middle| a \le x \le b, c \le y \le d \right\}$$

then

$$\iint\limits_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

Note: More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Fubini's Theorem

Exercises:
Calculate each double integral. Is one order of integration more straightforward than the other?

(a) #30. $\iint_R \frac{\tan\theta}{\sqrt{1-t^2}}\,dA, \quad R = \left\{(\theta, t) \,\middle|\, 0 \le \theta \le \frac{\pi}{3}, 0 \le t \le \frac{1}{2}\right\}$

***** Please work through the next example on your own. Solutions will be posted in the Solutions section of our OneNote Notebook *****

(b) #32. $\iint_R \frac{x}{1+xy}\,dA, \; R = [0,1]\times[0,1]$