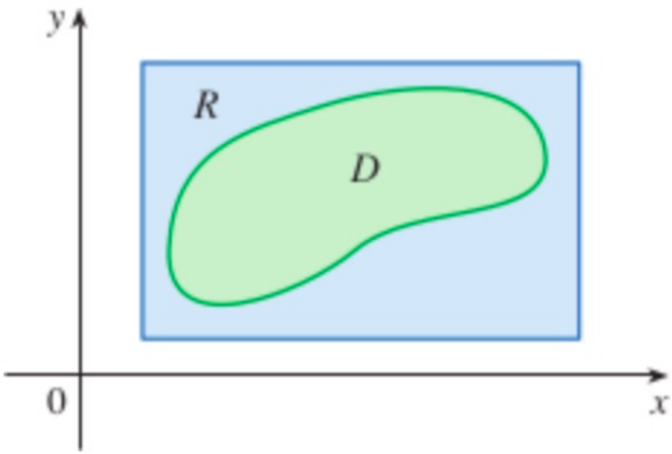
Double Integrals over General Regions

Section 15.2

General Regions

A general region *D* is said to be **bounded** if it can be enclosed in a rectangular region *R*.

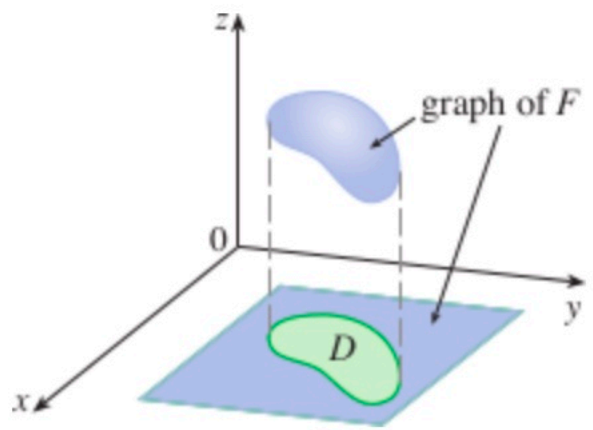


How do we compute $\iint_D f(x,y)\,dA$?

The Function F(x,y)

Define a new function *F,* with domain *R* as follows:

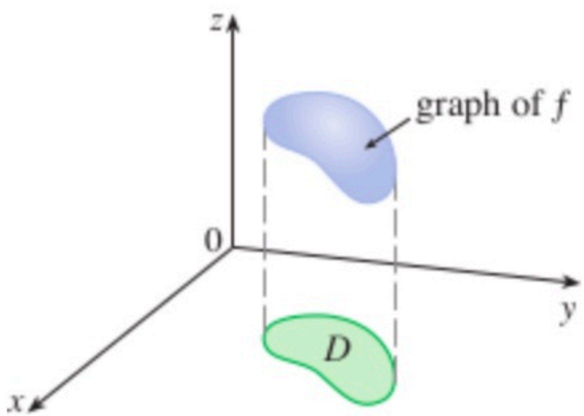
$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases}$$

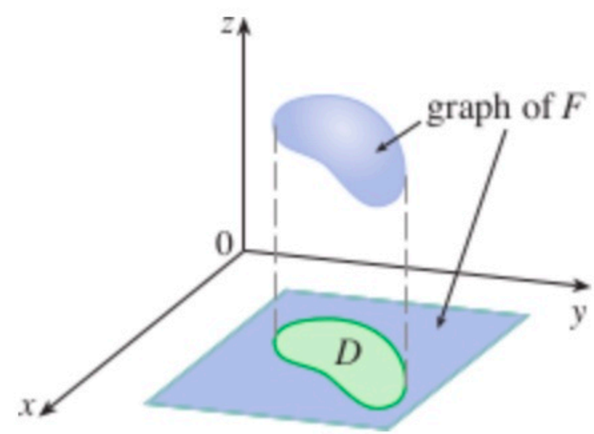


Defining the Double Integral over a General Region

If F is integrable over R, then we define the **double integral of *f* over *D*** by:

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$$



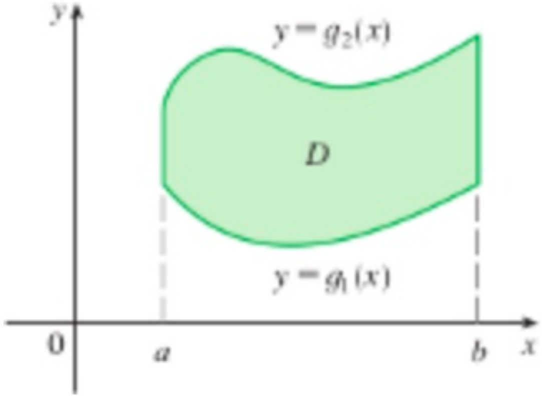


Type I Regions

A plane region D is said to be **type I** if it lies between the graphs of two continuous functions of x, that is

$$D = \left\{ (x,y) \middle| a \le x \le b,\ g_1(x) \le y \le g_2(x) \right\}$$

where g_1 and g_2 are continuous on $[a,b]$.



Double Integrals over Type I Regions

If *f* is continuous on a type I region *D* described by

$$D = \left\{ (x, y) \middle| a \le x \le b,\ g_1(x) \le y \le g_2(x) \right\}$$

then

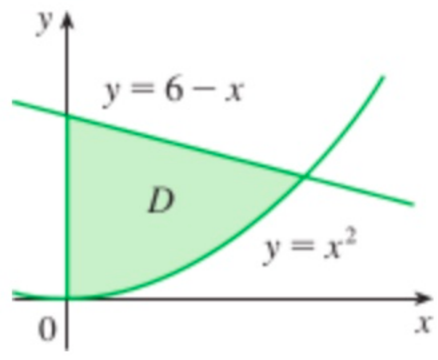
$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Note: When we evaluate the inner integral, we view *x* as being constant not only in *f(x,y)*, but also in the limits of integration $g_1(x)$ and $g_2(x)$.

Exercise

Evaluate the double integral.

#10. $\iint\limits_D xy\,dA$

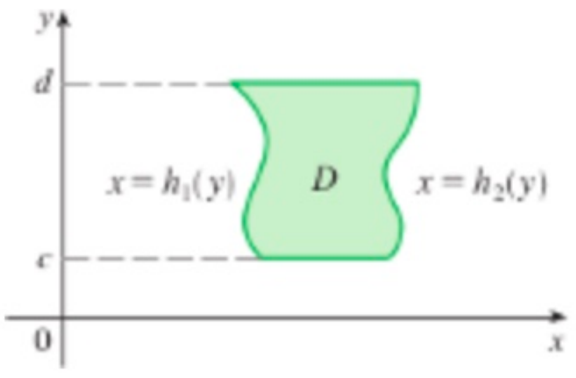


Type II Regions

A plane region D is said to be **type II** if it lies between the graphs of two continuous functions of y, that is

$$D = \left\{ (x, y) \middle| c \le y \le d,\ h_1(y) \le x \le h_2(y) \right\}$$

where h_1 and h_2 are continuous on [c,d].



Double Integrals over Type II Regions

If f is continuous on a type II region D described by

$$D = \left\{ (x, y) \middle| c \le y \le d,\ h_1(y) \le x \le h_2(y) \right\}$$

then

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Note: When we evaluate the inner integral, we view y as being constant not only in $f(x,y)$, but also in the limits of integration $h_1(y)$ and $h_2(y)$.

Exercise

Evaluate the double integral by treating *D* as a type II region.

#20. $\iint\limits_D y^2 e^{xy}\,dA, \quad D$ is bounded by $y = x, y = 4, x = 0$

Changing the Order of Integration

It may be helpful (or even necessary) to change the order of integration (justified by Fubini's Theorem) when faced with an iterated integral that is difficult to evaluate.

Exercise

Sketch the region of integration and change the order of integration.

#58. $\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} f(x,y)\,dx\,dy$

Exercise

Evaluate the integral by reversing the order of integration.

#64. $\int_0^2 \int_{y/2}^1 y\cos(x^3 - 1)\,dx\,dy$

Properties of Double Integrals

Assume that all of the following integrals exist.

5. $$\iint_D [f(x,y)+g(x,y)]\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$$

6. $$\iint_D cf(x,y)\,dA = c\iint_D f(x,y)\,dA$$ where *c* is a constant

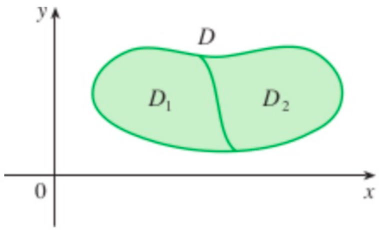
7. If $f(x,y) \geq g(x,y)$ for all (x,y) in D, then

$$\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA$$

Properties of Double Integrals

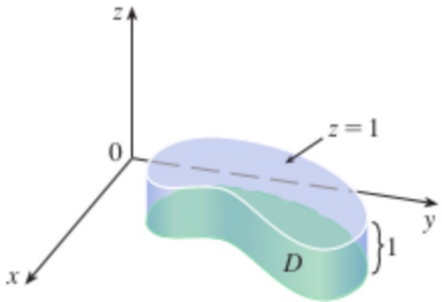
Assume that all of the following integrals exist.

8. If $D = D_1 \cup D_2$ and D_1 and D_2 do not overlap (except potentially on their boundaries), then



$$\iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA$$

9. $\iint_D 1dA = A(D)$

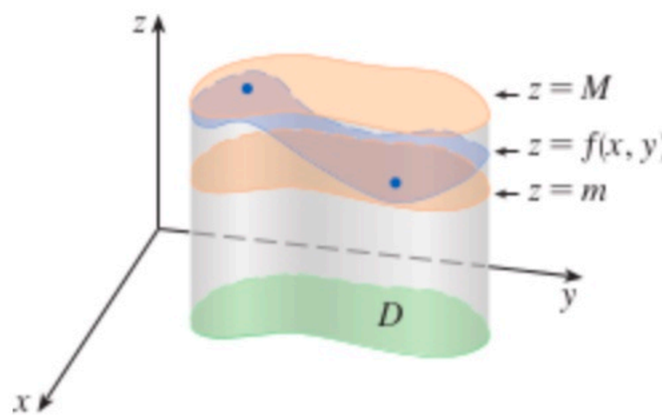


Properties of Double Integrals

Assume that all of the following integrals exist.

10. If $m \le f(x,y) \le M$ for all (x,y) in D, then

$$m \cdot A(D) \le \iint_D f(x,y)\,dA \le M \cdot A(D)$$



Exercise

#10. (revisited)

Express the region D as a type II region. Set up, but do not evaluate, an iterated integral for f and the region D. What property do you need to use to accomplish this?

$$\iint_D xy \, dA$$