MATHEMATICS 1LT3   SAMPLE FINAL EXAMINATION

Day Class
Duration of Examination: 3 hours
McMaster University

E. Clements

FIRST NAME (PRINT CLEARLY): ____________________

FAMILY NAME (PRINT CLEARLY): ____________________

Student No.: ____________________

THIS EXAMINATION PAPER HAS 18 PAGES AND 12 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF THE INVIGILATOR.

Note: Page 18 contains a partial table of values for $F(z)$.

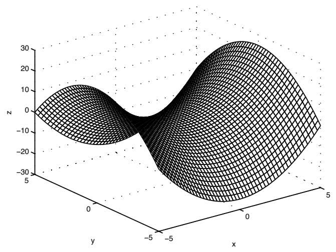
Total number of points is 80. Marks are indicated next to the problem number. You may use the McMaster standard calculator, Casio fx991 MS+. Write your answers in the space provided. EXCEPT ON QUESTION 1, YOU MUST SHOW WORK TO OBTAIN FULL CREDIT. **Good luck!**

Problem	Points	Mark
1	20	
2	6	
3	5	
4	4	
5	3	
6	4	
7	5	
8	7	
9	7	
10	6	
11	7	
12	6	
TOTAL	80	

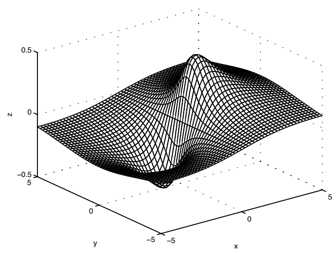
Question 1: For part (a), write the letter corresponding to the graph of the function next to the equation in the space provided. For parts (b)-(j), clearly circle the one correct answer.

1. (a) [2] Match the equation of each function with its graph below.

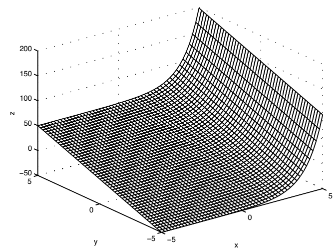
$f(x,y) = e^x + 10y$ ____ $\qquad g(x,y) = x^2 - y^2$ ____ $\qquad h(x,y) = \dfrac{x}{x^2+y^2+1}$ ____



(A)



(B)



(C)

(b) [2] In the basic model for the spread of a disease, $\dfrac{dI}{dt} = \alpha I(1-I) - \mu I$ where $\alpha, \mu > 0$, which of the following statements are true?

(I) $I^* = 0$ is a stable equilibrium.
(II) If $\mu > \alpha$, then the disease will eventually die out.
(III) If $\mu < \alpha$ and $I(0) > 0$, then $I(t) \to 1$ as $t \to \infty$.

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

(c) [2] A partial table of values for a function $f(x, y)$ is given below. Which of the following are positive?

(I) $f(4,1)$ (II) $f_x(4,1)$ (III) $f_{xx}(4,1)$

	$x=3$	$x=4$	$x=5$	$x=6$
$y=0$	2.3	2.2	2.0	1.7
$y=1$	2.4	2.5	2.7	3.0
$y=2$	2.5	2.7	2.9	3.2
$y=3$	2.6	3.0	3.0	3.3

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

(d) [2] The linearization of $f(x, y) = xe^{xy}$ at $(1, 0)$ is

(A) x (B) $-x$ (C) $y - x$ (D) $x + y$
(E) $x - y$ (F) y (G) $2x + y$ (H) 0

(e) [2] Various surveys have found that about 95% of claims that certain products are "green" (or "ecofriendly" or "organic") are either misleading or not true at all. Suppose that you buy 20 products that claim to be "green". Which of the following statements are true?

(I) The expected number of truly "green" products is 1.

(II) The probability that none of the products are truly "green" is 0.3585.

(III) The probability that all of the products are truly "green" is 0.00001935.

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

(f) [2] Let X count the number of heads obtained after three tosses of a fair coin. Which of the following statements are true?

(I) $E(X) = 1.5$

(II) $P(X \geq 1) = 0.875$

(III) $F(2) = 0.875$, where $F(x)$ is the cumulative distribution function of X.

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

(g) [2] Consider a population of school aged children comprised of 160 girls and 145 boys. Suppose that 3% of girls and 5% of boys within this population are estimated to be affected by ADHD. What is the probability that a randomly chosen child will be affected by ADHD?

(A) 0.03951	(B) 0.3248	(C) 0.3333	(D) 0.4051
(E) 0.1883	(F) 0.1286	(G) 0.1205	(H) 0.0421

(h) [2] Amoung women aged 40-50, the prevalence of breast cancer is 0.8%. A test for the presence of breast cancer (a mammogram, for example) shows a positive result in 90% of women who have breast cancer and in 5% of women who do not have breast cancer. Suppose that a woman in this age group tests positive for breast cancer. What is the probability (approximately) that she actually has it?

(A) 0.233	(B) 0.768	(C) 0.685	(D) 0.562
(E) 0.148	(F) 0.865	(G) 0.921	(H) 0.127

(i) [2] Certain types of a rare strain of respiratory infection occur in about 3 out of 2,000 people. During a particularly bad flu season, 12 out of 5,000 people were diagnosed with the infection. What is the probability of this event occurring?

(A) 0.036575	(B) 0.048574	(C) 0.037425	(D) 0.133589
(E) 0.046471	(F) 0.865751	(G) 0.0016575	(H) 0.0055238

(j) [2] Suppose that $X \sim N(5, 2^2)$. Which of the following statements is/are true?

(I) $P(1 \leq X \leq 9) \approx 0.955$

(II) $P(X > 4) \approx 0.691$

(III) $P(X \leq x) = 0.8$ when $x \approx 6.7$

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

Questions 2-12: You must show work to obtain full credit.

2. State whether each statement is **true or false** and then **explain** your reasoning.

(a) [2] $x^* = 0$ is a stable equilibrium of the autonomous differential equation $\dfrac{dx}{dt} = 1 - e^x$.

(b) [2] The range of $g(x, y) = e^{x^2+y^2}$ is $(0, \infty)$.

(c) [2] Suppose that $\nabla f(2,3) = 4\mathbf{i} - \mathbf{j}$. Then $D_{\mathbf{u}} f(2,3) = 5$ for some direction $\mathbf{u}$.

3. A population of ladybugs changes according to the logistic differential equation

$$\frac{dL}{dt} = 0.05L\left(1 - \frac{L}{200}\right)$$

(a) [2] Graph the rate of change, $\frac{dL}{dt}$, as a function of L. Clearly label intercepts.

(b) [2] Draw a phase-line diagram for this differential equation.

(c) [1] Suppose that the population will die out if the number of ladybugs drops below 30. Write a new differential equation (i.e., modify the one above) to reflect this observation.

4. The following pair of equations represent the population growth of two different species where one is the predator, the other is the prey and t is measured in months.

$$\frac{dx}{dt} = 0.4x - 0.001xy \qquad \frac{dy}{dt} = -0.01y + 0.0002xy$$

(a) [2] State which variable represents the prey population and explain why.

(b) [2] Suppose that $x_0 = 500$ and $y_0 = 80$. Using Euler's Method with a step size of one month, estimate the size of population x two months from now.

5. [3] Solve the separable equation $\frac{dP}{dt} = \frac{2tP}{1+t^2}$, with initial condition $P(0) = 1$.

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Student No.: ____________________

6. Consider the function $g(x, y) = \ln(xy)$.

(a) [2] Find and sketch the domain of g.

(b) [2] Create a contour map for g including level curves for $k = -1$, $k = 0$, and $k = 1$.

7. (a) [2] Describe what is meant by $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

(b) [3] Show that $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{5x^4 + y^2}$ does not exist. Sketch the domain of the function and the paths you've chosen to approach $(0,0)$ along.

8. Consider the function $f(x, y) = (2x - y + 5)^{\frac{3}{2}}$.

(a) [2] Compute f_x and f_y. Find and sketch their domains.

(b) [2] A sufficient condition for differentiability states that if f_x and f_y are continuous on an open disk $B_r(a, b)$ then f is differentiable at (a, b). Can we use this to determine if f is differentiable at $(1, 7)$? Explain why or why not.

(c) [3] Compute the directional derivative of $f(x, y) = (2x - y + 5)^{\frac{3}{2}}$ at the point $(0, 1)$ in the direction specified by $\mathbf{v} = 2\mathbf{i} + \mathbf{j}$.

9. (a) [2] Show that $(1,1)$ is the only critical point of $f(x,y) = x + y + \dfrac{1}{xy}$.

(b) [3] Use the second derivatives test to determine whether f has a local maximum, a local minimum, or a saddle point at $(1,1)$.

(c) [2] Without using the second derivatives test, determine whether $(0,0)$ corresponds to a local maximum, local minimum, or saddle point of the function $g(x,y) = x^3 - 2y^2 + 3xy + 1$.

10. A population of bears is modelled by $b_{t+1} = b_t + I_t$, where b_t represents the number of bears in year t. Suppose that the immigration term is $I_t = 8$ with a 40% chance and $I_t = 0$ with a 60% chance. Assume that $b_0 = 40$ and that immigration from year to year is independent.

(a) [3] Let X count the number of bears after 2 years. Find the expected value and standard deviation of X.

(b) [3] What is the probability that there will be more than 100 bears after 10 years? (Hint: Let N count the number of years immigration occurs and use the Binomial Distribution.)

11. Suppose that the lifetime of a tree is given by the probability density function $f(t) = 0.01e^{-0.01t}$, where t is measured in years, $0 \leq t < \infty$.

(a) [2] Determine the cumulative distribution function, $F(t)$.

(b) [2] What is the probability that the tree will live longer than 70 years?

(c) [3] Find the average lifetime of the tree. (Recall: $\int u\, dv = uv - \int v\, du$.)

12. The wingspan, W, of a blue jay is normally distributed with a mean of 39 cm and a standard deviation of 3 cm.

(a) [2] Sketch the graph of the probability density function for W, labelling the mean, maximum, and location of inflection points.

(b) [2] What is the probability that a randomly chosen blue jay has a wingspan wider than 42 cm?

(c) [2] Using substitution and the fact that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$, show that $\int_{-\infty}^{\infty} f(w)\,dw = 1$ (this, along with your graph in part (a), verifies that $f(w)$ is a valid probability density function).

ROUGH WORK

Partial Table of Values for $F(z)$ where $Z \sim N(0,1)$

z	$F(z)$	z	$F(z)$	z	$F(z)$	z	$F(z)$
0	0.500000	1	0.841345	2	0.977250	3	0.998650
0.05	0.519938	1.05	0.853141	2.05	0.979818	3.05	0.998856
0.1	0.539828	1.1	0.864334	2.1	0.982136	3.1	0.999032
0.15	0.559618	1.15	0.874928	2.15	0.984222	3.15	0.999184
0.2	0.579260	1.2	0.884930	2.2	0.986097	3.2	0.999313
0.25	0.598706	1.25	0.894350	2.25	0.987776	3.25	0.999423
0.3	0.617911	1.3	0.903200	2.3	0.989276	3.3	0.999517
0.35	0.636831	1.35	0.911492	2.35	0.990613	3.35	0.999596
0.4	0.655422	1.4	0.919243	2.4	0.991802	3.4	0.999663
0.45	0.673645	1.45	0.926471	2.45	0.992857	3.45	0.999720
0.5	0.691462	1.5	0.933193	2.5	0.993790	3.5	0.999767
0.55	0.708840	1.55	0.939429	2.55	0.994614	3.55	0.999807
0.6	0.725747	1.6	0.945201	2.6	0.995339	3.6	0.999840
0.65	0.742154	1.65	0.950529	2.65	0.995975	3.65	0.999869
0.7	0.758036	1.7	0.955435	2.7	0.996533	3.7	0.999892
0.75	0.773373	1.75	0.959941	2.75	0.997020	3.75	0.999912
0.8	0.788145	1.8	0.964070	2.8	0.997445	3.8	0.999928
0.85	0.802337	1.85	0.967843	2.85	0.997814	3.85	0.999941
0.9	0.815940	1.9	0.971283	2.9	0.998134	3.9	0.999952
0.95	0.828944	1.95	0.974412	2.95	0.998411	3.95	0.999961
						4	0.999968