MATHEMATICS 1LT3 TEST 2

Evening Class
Duration of Test: 60 minutes
McMaster University

E. Clements

25 July 2017

FIRST NAME (please print): Soln S

FAMILY NAME (please print): ______________

Student No.: ______________

THIS TEST HAS 8 PAGES AND 8 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number in square brackets. You may use the McMaster standard calculator, Casio fx991 MS+, on this test.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL, YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit, except for Question 1.

Problem	Points	Mark
1	6	
2	6	
3	4	
4	4	
5	6	
6	6	
7	4	
8	4	
TOTAL	40	

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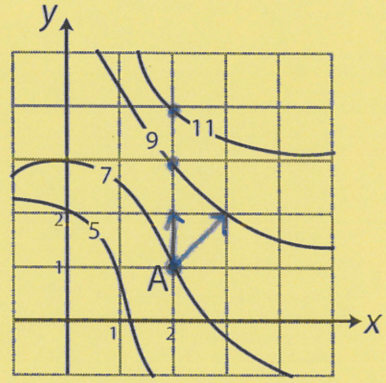
1. **Multiple Choice**. Clearly **circle** the one correct answer.

(a) [3] Determine which of the following is/are true for the function $f(x, y)$ whose contour map is given below.

(I) $f_y(2,1) > 0$ ✓ (II) $f_{yy}(2,1) > 0$ ✓ (III) $D_{\mathbf{v}}f(2,1) \approx 2$ when $\mathbf{v} = \mathbf{i} + \mathbf{j}$ ✗



$D_v f(2,1) \approx \frac{9-7}{\sqrt{2}}$
$\approx \frac{2}{\sqrt{2}}$.
$\approx \sqrt{2}$

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

(b) [3] Consider the random experiment of rolling a fair, six-sided die. Let A be the event of rolling an even number and let B be the event of rolling a number greater than 2. Which of the following is/are true?

(I) $P(A \cap B) = \frac{1}{3}$ ✓ (II) $P(A \cup B) = \frac{5}{6}$ ✓ (III) $P(A^C) = \frac{1}{6}$ ✗

$A = \{2, 4, 6\}$ $B = \{3, 4, 5, 6\}$
$A \cap B = \{4, 6\}$ ... $P(A \cap B) = \frac{2}{6} = \frac{1}{3}$
$A \cup B = \{2, 3, 4, 5, 6\}$ ... $P(A \cup B) = \frac{5}{6}$
$A^c = \{1, 3, 5\}$ ... $P(A^c) = \frac{3}{6} = \frac{1}{2}$

(A) none (B) I only (C) II only (D) III only
(E) I and II (F) I and III (G) II and III (H) all three

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2. State whether each statement is **true or false**. **Explain** your reasoning.

(a) [2] The equation of the tangent plane to the graph of $f(x, y) = e^y \sin x$ at $(\pi/2, 0)$ is $z = y + 1$.

$f(\frac{\pi}{2}, 0) = e^0 \sin \frac{\pi}{2} = 1$

$f_x = e^y \cos x \ldots \quad f_x(\frac{\pi}{2}, 0) = e^0 \cos \frac{\pi}{2} = 0$

$f_y = e^y \sin x \ldots \quad f_y(\frac{\pi}{2}, 0) = e^0 \sin \frac{\pi}{2} = 1$

$\therefore$ eq^N of the tangent plane is $z = 1 + 0(x - \frac{\pi}{2}) + 1(y - 0) = 1 + y$

$\therefore$ TRUE

(b) [2] The maximum rate of increase of the function $g(x, y) = \ln(x/y)$ at $(1, 1)$ is 0.5.

$g_x = \frac{1}{\frac{x}{y}} \cdot \left(\frac{1}{y}\right) \ldots \quad g_x(1,1) = 1$

$g_y = \frac{1}{\frac{x}{y}} \cdot \left(-\frac{x}{y^2}\right) \ldots \quad g_y(1,1) = -1$

$\nabla g(1,1) = \hat{i} - \hat{j}$

max. rate of change of g at $(1,1) = \|\nabla g(1,1)\|$

$= \sqrt{1^2 + (-1)^2}$

$= \sqrt{2}$

$\therefore$ FALSE

(c) [2] Suppose that a quiz has five multiple-choice questions, each with three choices. If a student randomly answers all questions, then the probability that they will answer at least one question correctly is about 0.86831.

A = at least one correctly

A^c = none are correct

A_i = i^{th} question correct

$P(A_i) = \frac{1}{3} \quad P(A_i^c) = \frac{2}{3}$

$P(A) = 1 - P(A^c)$

$= 1 - P\left(\bigcap_{i=1}^{5} A_i^c\right)$

$= 1 - \prod_{i=1}^{5} P(A_i^c)$

$= 1 - \left(\frac{2}{3}\right)^5$

$\doteq 0.86831$

$\therefore$ TRUE

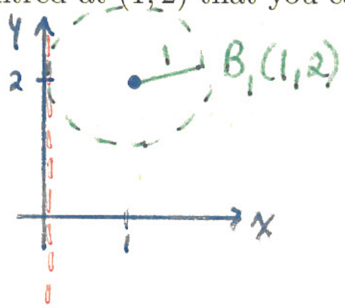
3. Consider the function $f(x, y) = y\sqrt{x} + y^3$. $x \geq 0$

(a) [2] Compute the partial derivatives $f_x(x, y)$ and $f_y(x, y)$. State the domain of each.

$f_x = \frac{y}{2\sqrt{x}}$ $\quad$ $f_y = \sqrt{x} + 3y^2$

$x > 0$ $\quad$ $x \geq 0$

(b) [2] Explain why $f(x, y) = y\sqrt{x} + y^3$ is differentiable at $(1, 2)$. What is the largest open disk centred at $(1, 2)$ that you can use?



Since f_x and f_y are continuous ⊛ on $B_1(1,2)$, f is differentiable at $(1,2)$

Largest OPEN disk centred at $(1,2)$ has radius 1.

⊛ f_x and f_y are algebraic functions and thus continuous on their domain

4. [4] Compute the directional derivative of the function $g(x, y) = \arctan(3x + y)$ at the point $(0, 1)$ in the direction specified by $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$.

$\|\vec{v}\| = \sqrt{4^2 + (-3)^2} = 5$

$\hat{u} = \frac{4}{5}\hat{i} - \frac{3}{5}\hat{j}$

$g_x = \frac{1}{1+(3x+y)^2}(3) \ldots g_x(0,1) = \frac{3}{2}$

$g_y = \frac{1}{1+(3x+y)^2}(1) \ldots g_y(0,1) = \frac{1}{2}$

$D_{\vec{v}}\, g(0,1) = \frac{3}{2}\left(\frac{4}{5}\right) + \frac{1}{2}\left(-\frac{3}{5}\right)$

$= \frac{9}{10}$

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5. Consider the function $f(x, y) = x^3 - 2y^2 + 3xy + 1$.

(a) [3] Find the critical points of $f(x, y)$.

$f_x = 3x^2 + 3y = 3(x^2+y) \qquad f_y = -4y + 3x$

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} x^2+y=0 \\ -4y+3x=0 \end{cases} \Rightarrow \begin{cases} y = -x^2 & ① \\ y = \frac{3}{4}x & ② \end{cases}$$

sub ② into ①: $\frac{3}{4}x = -x^2$

$\Rightarrow x\left(x + \frac{3}{4}\right) = 0$

$\Rightarrow x = 0$ or $x = \frac{-3}{4}$

sub $x=0$ into ②: $y = 0 \Rightarrow (0,0)$ is a critical point

sub $x = -\frac{3}{4}$ into ②: $y = \frac{-9}{16} \Rightarrow \left(-\frac{3}{4}, -\frac{9}{16}\right)$ is a critical point

(b) [3] Using the second derivatives test, classify the critical points from part (a).

$f_{xx} = 6x \quad f_{yy} = -4 \quad f_{xy} = f_{yx} = 3$

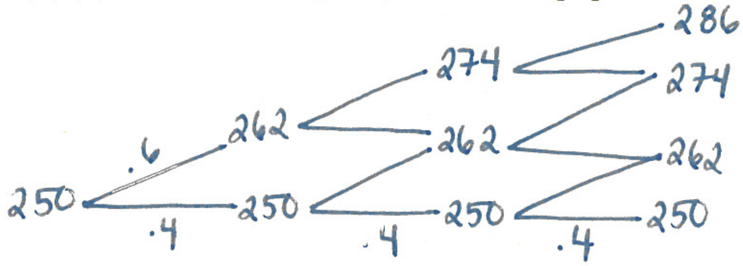
$\therefore D = 6x(-4) - 3^2 = -24x - 9$

$D(0,0) = -9 \Rightarrow (0,0)$ is a saddle point

$$\left.\begin{array}{l} D\left(-\frac{3}{4}, \frac{-9}{16}\right) = -24\left(-\frac{3}{4}\right) - 9 = 9 \\ f_{yy}\left(-\frac{3}{4}, -\frac{9}{16}\right) = -4 \end{array}\right\} \Rightarrow f \text{ has a local max. at } \left(-\frac{3}{4}, -\frac{9}{16}\right).$$

6. Consider a population of 250 moose. Suppose that within any given year, there is a 60% chance that the population will increase by 12 and a 40% chance that it will stay the same.

(a) [2] Write the sample space for the population of moose after 3 years.



$S = \{250, 262, 274, 286\}$

(b) [2] What is the probability that the population will have increased after 3 years?

$A = P_3 > P_0 \qquad A^c = P_3 \leq P_0 = \{250\}$.

$$\begin{aligned} P(A) &= 1 - P(A^c) \\ &= 1 - P(\{250\}) \\ &= 1 - (0.4)^3 \\ &\approx 0.936 \end{aligned}$$

(c) [2] Suppose that conditions changed and now within any given year, there is a 75% chance that the population will increase by 12 and a 25% it will decrease by 20. What is more likely to happen to the number of moose over time? A net increase or a decrease? Explain.

Consider the pop^N dynamics over a 4-year period.
In 3 years, we'd expect an increase of 12
In 1 year, we'd expect a decrease of 20.
Net moose in 4 years: $3 \times 12 - 1 \times 20 = 16$

$\therefore$ The pop^N is likely to increase over time (by 4 moose per year, on average).

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7. Consider the random experiment of rolling two, fair six-sided dice.

(a) [2] Find the probability that the sum is 7.

$|S| = 36$

A = sum is 7 $= \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

$$P(A) = \frac{6}{36} = \frac{1}{6}$$

(b) [2] Using *conditional probability*, find the probability that the sum is 7 given that one die shows a number larger than 3.

C = one die larger than 3

(*) $A \cap C = A$

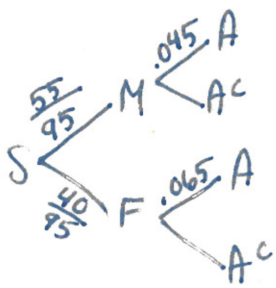
$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{\frac{6}{36}}{\frac{18}{36}} = \frac{6}{18} = \frac{1}{3}$$

(or C = at least one die > 3

Then $P(C) = \frac{27}{36}$ and $P(A|C) = \frac{6}{27} = \frac{2}{9}$)

8. The incidence of asthma in young adults is 6.4% for females and 4.5% for males. Consider a group of young adults consisting of 40 females and 55 males.

(a) [2] What is the probability that a randomly chosen young adult from this group has asthma?



$$P(A) = 0.045\left(\frac{55}{95}\right) + 0.065\left(\frac{40}{95}\right)$$

$$\approx 0.053$$

(b) [2] What is the probability that a young adult from this group who has asthma is female?

$$P(F|A) = \frac{P(A|F)P(F)}{P(A)}$$

$$= \frac{0.065\left(\frac{40}{95}\right)}{.045\left(\frac{55}{95}\right) + .065\left(\frac{40}{95}\right)}$$

$$\approx 0.51$$