MATHEMATICS 1LT3 TEST 1

Evening Class
Duration of Test: 60 minutes
McMaster University

E. Clements

6 July 2017

FIRST NAME (please print): Sol^Ns

FAMILY NAME (please print): __________

Student No.: __________

THIS TEST HAS 8 PAGES AND 8 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Total number of points is 40. Marks are indicated next to the problem number in square brackets. You may use the McMaster standard calculator, Casio fx991 MS+, on this test.

USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL, YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

You need to show work to receive full credit, except for Question 1.

Problem	Points	Mark
1	9	
2	6	
3	8	
4	5	
5	3	
6	3	
7	3	
8	3	
TOTAL	40	

Name:____________________

Student No.: ________________

---

1. For parts (a) and (b), clearly circle the one correct answer. For part (c), write the letter corresponding to the graph of the function next to the equation in the space provided.

(a) [3] Consider the model for the spread of a disease, $dI/dt = \overset{\alpha}{0.4}I(1-I) - \overset{\mu}{0.1}I$, where I represents the proportion of infected individuals in the population. Which of the following statements is/are true?

$\alpha > \mu$

(I) There are two biologically plausible equilibria. ✓ $I_1^* = 0$

(II) If initially 2% of the population is infected, then I will decrease. X $I_2^* = 1 - \frac{\mu}{\alpha} = 0.75$

(III) If initially 80% of the population is infected, then I will decrease. ✓

0 .75 1

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

(b) [3] Which of the following statements is/are true for the predator-prey equations, $dx/dt = 0.2x - 0.01xy$ and $dy/dt = -0.3y + 0.001xy$.

(I) The variable x represents the prey population. ✓

(II) The per capita growth rate of y is a decreasing function. X $\frac{y'}{y} = \underbrace{-0.3 + 0.001x}_{\text{increasing}}$

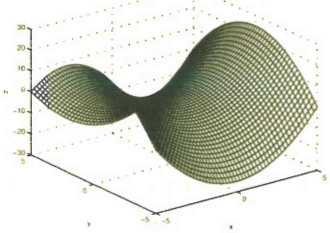
(III) $(x, y) = (300, 20)$ is an equilibrium solution. ✓

$\left.\frac{dx}{dt}\right|_{x=300, y=20} = 0$ $\left.\frac{dy}{dt}\right|_{x=300, y=20} = 0$

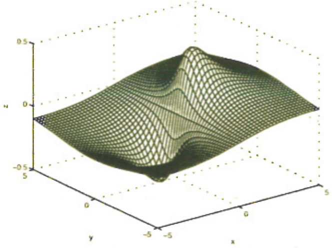
(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

(c) [3] Match the equation of each function with its graph below.

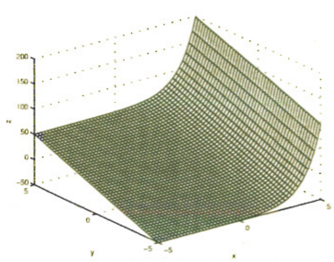
$f(x, y) = e^x + 10y$ __C__ $g(x, y) = x^2 - y^2$ __A__ $h(x, y) = \dfrac{x}{x^2 + y^2 + 1}$ __B__



(A)



(B)



(C)

Name:

Student No.:

2. State whether each statement is **true or false**. **Explain** your reasoning.

(a) [2] $y^* = 0$ is a stable equilibrium of $dy/dt = y^3 - y$.

$f(y) = y(y^2-1)$

$\frac{dy}{dt} = 0$ when $\underbrace{y=0 \text{ or } y=\pm 1}_{\text{equilibria}}$

$f'(y) = 3y^2 - 1$

$f'(0) = -1 \Rightarrow y^* = 0$ is a stable equilibruim

$\therefore$ TRUE

(b) [2] The range of $z = e^{\sqrt{1-x^2-y^2}}$ is $(0, e]$.

$1-x^2-y^2 \geq 0$

$x^2+y^2 \geq 0$

$-x^2-y^2 \leq 0$

$0 \leq 1-x^2-y^2 \leq 1$

$0 \leq \sqrt{1-x^2-y^2} \leq \sqrt{1}$

$e^0 \leq e^{\sqrt{1-x^2-y^2}} \leq e^1$

$\Rightarrow$ the range is $[1, e]$.

$\therefore$ FALSE

(c) [2] The function $g(x,y) = \begin{cases} \dfrac{\cos(xy)}{x^2+y^2+1} & \text{if} \quad (x,y) \neq (0,0) \\ 1 & \text{if} \quad (x,y) = (0,0) \end{cases}$ is continuous at $(0,0)$.

① $\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} \frac{\cos(xy)}{x^2+y^2+1} = \frac{\cos 0}{1} = 1$

② $g(0,0) = 1$

since $\lim_{(x,y)\to(0,0)} g(x,y) = g(0,0)$, g is continuous at $(0,0)$.

def^N of continuity

$\therefore$ TRUE

Name:____________________
Student No.: ________________

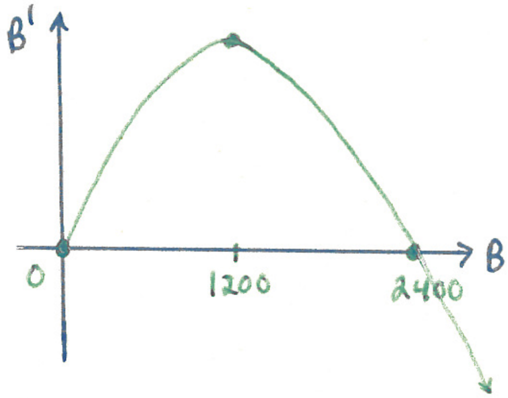
3. A population of birds is modelled by $\dfrac{dB}{dt} = 0.8B\left(1 - \dfrac{B}{2400}\right)$.

(a) [2] Find the equilibria of this equation. What does the larger equilibrium represent?

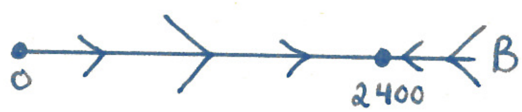
$\dfrac{dB}{dt} = 0$ when $B = 0$ or $B = 2400$.

The carrying capacity (max. population environment is capable of sustaining in the long run) is 2400 birds.

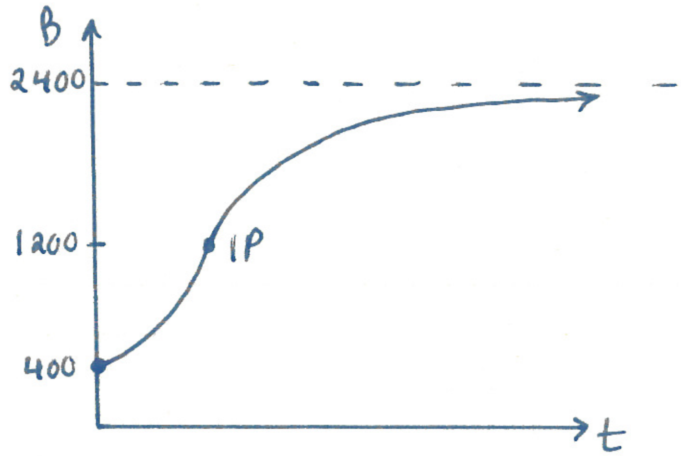
(b) [2] Graph $\dfrac{dB}{dt}$ as a function of B.



(c) [2] Draw a phase-line diagram for $\dfrac{dB}{dt} = 0.8B\left(1 - \dfrac{B}{2400}\right)$.



(d) [2] Suppose that initially there are 400 birds. Sketch the solution curve $B(t)$.



Name:________________
Student No.: ________________

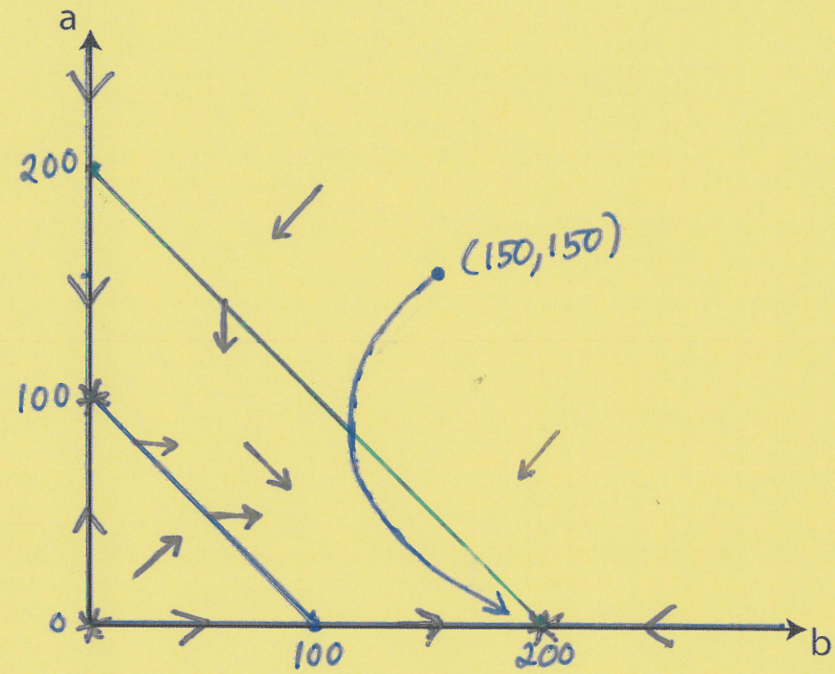
4. Consider the competition equations

$$\frac{da}{dt} = 2\left(1 - \frac{a+b}{100}\right)a, \quad \frac{db}{dt} = 2\left(1 - \frac{a+b}{200}\right)b$$

(a) [2] Find and graph the nullclines in the phase plane.

$\frac{da}{dt} = 0 \Rightarrow a = 0$ or $a = 100 - b$

$\frac{db}{dt} = 0 \Rightarrow b = 0$ or $a = 200 - b$



(b) [1] Identify the equilibria.

(0,0), (200,0), (0,100)

(c) [2] Add direction arrows to your phase-plane diagram in part (a). Use the direction arrows to sketch a phase-plane trajectory starting from $a(0) = 150$ and $b(0) = 150$.

Name:______________

Student No.: ______________

5. [3] Use the separation of variables technique to solve $\frac{dy}{dx} = \frac{3y}{1+4x^2}$, where $y(0) = 5$.

$$\int \frac{1}{y}\,dy = \int \frac{3}{1+(2x)^2}\,dx$$

$$\ln|y| = \frac{3}{2}\arctan(2x) + C$$

$$|y| = e^{\frac{3}{2}\arctan(2x) + C}$$

$$y = \pm e^{C} e^{\frac{3}{2}\arctan(2x)}$$

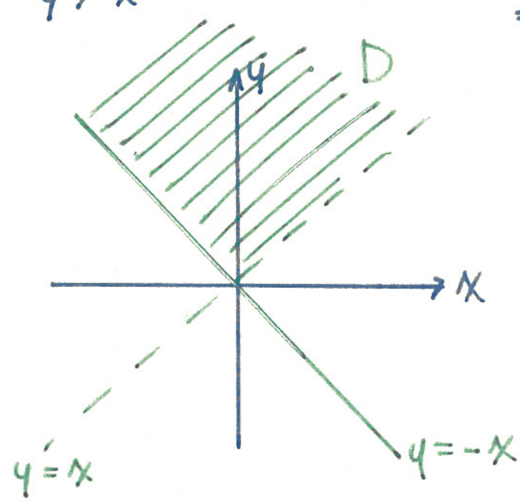
call this constant "A"

$$y(0)=5 \Rightarrow 5 = Ae^{\frac{3}{2}\arctan(0)} \Rightarrow 5 = Ae^{0} \Rightarrow A = 5$$

$$\therefore\ y(x) = 5e^{\frac{3}{2}\arctan(2x)}$$

6. [3] Find and sketch the domain of $f(x, y) = \ln(y - x) + \sqrt{y + x}$.

① $y - x > 0$ $\Rightarrow y > x$ and ② $y + x \geq 0$ $\Rightarrow y \geq -x$



Name:____________________
Student No.: ________________

7. [3] Create a contour map for $g(x, y) = 2\arctan(xy)$. Include level curves corresponding to $k = -2$, $k = 0$, and $k = 2$.

level curves:

$2\arctan(xy) = k$, where $k \in (-\frac{\pi}{2}, \frac{\pi}{2})$

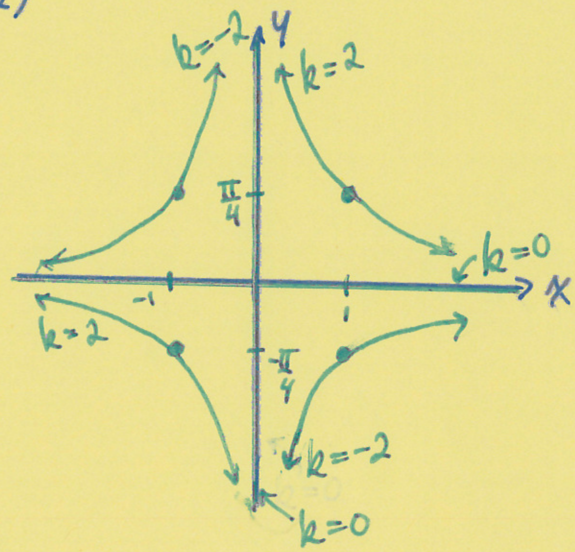
$\Rightarrow xy = \tan(\frac{k}{2})$

$\Rightarrow y = \dfrac{\tan(\frac{k}{2})}{x}$

$k=-2: \; y = \dfrac{-\frac{\pi}{4}}{x}$

$k=0: \; xy = 0 \Rightarrow x=0$ or $y=0$

$k=2: \; y = \dfrac{\frac{\pi}{4}}{x}$



8. [3] Show that $\displaystyle\lim_{(x,y)\to(0,0)} \underbrace{\frac{4xy^2}{3x^2+2y^4}}_{f}$ does not exist.

$f(0,y) = \dfrac{0}{2y^4} = 0$

$f(y^2,y) = \dfrac{4y^2 \cdot y^2}{3(y^2)^2+2y^4} = \dfrac{4y^4}{5y^4} = \dfrac{4}{5}$

since $f(x,y) \to 0$ as $(x,y)\to(0,0)$ along $x=0$
and $f(x,y) \to \frac{4}{5}$ as $(x,y)\to(0,0)$ along $x=y^2$

$\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ D.N.E.