Students' Solutions Manual

Functions of Several Variables

This manual contains solutions to odd-numbered exercises from the book *Functions of Several Variables* by Miroslav Lovrić, published by Nelson Publishing.

Keep in mind that the solutions provided represent *one* way of answering a question or solving an exercise. In many cases there are alternatives, so make sure that you don't dismiss your solution just because it does not look like the solution in this manual.

This solutions manual is not meant to be read! Think, try to solve an exercise on your own, investigate different approaches, experiment, see how far you get. If you get stuck and don't know how to proceed, try to understand why you are having difficulties before looking up the solution in this manual. If you just read a solution you might fail to recognize the hard part(s); even worse, you might completely miss the point of the exercise.

I accept full responsibility for errors in this text and will be grateful to anybody who brings them to my attention. Your comments and suggestions will be greatly appreciated.

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Section 1 Introduction

1. Comparing with the general form $z = ax+by+c$, we see that $f(x,y) = 3-x+6y$ is linear ($a=-1$, $b=6$, $c=3$); $h(x,y) = 3x-2y+4$ is linear ($a=3$, $b=-2$, $c=4$); and $k(x,y)=1$ is linear ($a=0$, $b=0$, $c=1$). The function $g(x,y) = 2xy$, is not linear, since it involves the product of x and y.

3. We can generate many examples by using the fact that the functions $\ln t$ and $1/\sqrt{t}$ are defined for $t>0$. For instance, the domain of the functions $f(x,y) = \ln x + \ln y$ (or $\ln x - \ln y$), $f(x,y) = 1/\sqrt{x} + 1/\sqrt{y}$, $f(x,y) = \ln x/\sqrt{y}$, and $f(x,y) = \ln(xy) + \ln x$ is the set $\{(x,y) \mid x>0 \text{ and } y>0\}$.

5. The distance from (x,y) to the origin is given by $d(x,y) = \sqrt{x^2+y^2}$. Thus,

$$f(x,y) = k \cdot \frac{1}{d^2(x,y)} = \frac{k}{x^2+y^2}$$

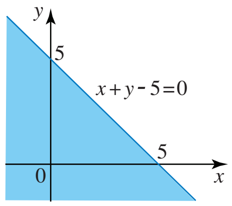
for a real number k. From $f(-1,2)=4$ we get that

$$4 = \frac{k}{(-1)^2+2^2} = \frac{k}{5}$$

and $k=20$. Thus, $f(x,y) = 20/(x^2+y^2)$.

7. The function f is defined whenever $x \neq 0$; i.e., it is defined at all points (x,y) where $x \neq 0$ and y can be any real number. Thus, the domain of f is the xy-plane without the y-axis.

9. A point (x,y) in the domain of f satisfies $5-x-y \geq 0$. The equation $5-x-y=0$, i.e., $x+y-5=0$, represents the line whose x- and y- intercepts are 5. This line divides the xy-plane into two halves, and we use a test point to figure out which half satisfies $5-x-y \geq 0$. Take the origin: since $5-(0)-(0) \geq 0$ is true, it follows that the domain of f is the half of the xy-plane which contains the origin (i.e., the left half) including the line that borders it (because of the equals sign in $5-x-y \geq 0$); see the figure below.



11. Since x^2+y^2-1 is in the denominator, the points (x,y) which are not in the domain of f satisfy $x^2+y^2-1=0$, i.e., $x^2+y^2=1$. Thus, the domain of f consists of all points in $\mathbb{R}^2$, except those that lie on the circle of radius 1 centred at the origin.

13. Since $D>0$, the term $1/\sqrt{4\pi Dt}$ is defined when $t>0$. The term $-x^2/4Dt$ in the exponent of e is defined as long as $t \neq 0$ (it is defined for all x). The domain of $c(x,t)$ consists of all points (x,t), where x is any real number and $t>0$.

15. If $a=b=0$, then $f(x,y)=c$ and so the range of f consists of a single value, $R=\{c\}$. If one of a or b is not zero, or if both $a \neq 0$ and $b \neq 0$, the range of f consists of all real numbers. To prove this, we pick a real number r and find x and y so that $f(x,y) = ax+by+c = r$.

If $a=0$ (in which case $b \neq 0$), then $by+c=r$ and $y=(r-c)/b$. In this case, for any x,

$$f(x,(r-c)/b) = 0 \cdot x + b \cdot \frac{r-c}{b} + c = 0 + r - c + c = r$$

If $b = 0$, then $ax + c = r$ and $x = (r - c)/a$. For any y,

$$f((r-c)/a, y) = a \cdot \frac{r-c}{a} + 0 \cdot y + c = r - c + 0 + c = r$$

If $a \neq 0$ and $b \neq 0$ then we have infinitely many choices for x and y which satisfy $ax + by + c = r$. Pick (the simplest value) $y = 0$; then $x = (r - c)/a$, and, as above, $f((r - c)/a, 0) = r$.

17. The function f is a polynomial and so its domain is $\mathbb{R}^2$. To prove that the range of f is $\mathbb{R}$, we pick any real number r and show that there is a point (x, y) such that $f(x, y) = r$. From $f(x, y) = x + 12 = r$, we get $x = r - 12$. Take the point $(r - 12, y)$, where y is any real number. Then $f(r - 12, y) = r - 12 + 12 = r$.

19. Since the domain of the function $h(x, y) = x^2 + y^2$ is $\mathbb{R}^2$, and the exponential function e^t is defined for all real numbers t, we conclude that the domain of the composition $g(x, y) = e^{x^2+y^2}$ is $\mathbb{R}^2$.

Since $x^2 + y^2 \geq 0$, we conclude that

$$g(x, y) = e^{x^2+y^2} \geq e^0 = 1.$$

Thus, the range of g is a subset of $[1, \infty)$. Now we show that the range is equal to $[1, \infty)$: pick a number r in $[1, \infty)$; then

$$\begin{aligned} g(x, y) = e^{x^2+y^2} &= r \\ x^2 + y^2 &= \ln r \end{aligned}$$

Take $y = 0$, so that $x^2 = \ln r$ and $x = \pm\sqrt{\ln r}$. We are done, since

$$f(\pm\sqrt{\ln r}, 0) = e^{(\pm\sqrt{\ln r})^2+0^2} = e^{\ln r} = r$$

21. Because the polynomial under the square root satisfies $2 + x^2 + 5y^2 \geq 0$ for all x, y, the domain of g is $\mathbb{R}^2$.

From $x^2 + 5y^2 \geq 0$ we conclude that $2 + x^2 + 5y^2 \geq 2$ and thus

$$g(x, y) = \sqrt{2 + x^2 + 5y^2} \geq \sqrt{2}$$

So, the range R of g is a subset of $[\sqrt{2}, \infty)$. We show that $R = [\sqrt{2}, \infty)$ by proving that for any $r \in [\sqrt{2}, \infty)$ there exists a point (x, y) so that $f(x, y) = r$. Let $r \in [\sqrt{2}, \infty)$. Then

$$\begin{aligned} g(x, y) = \sqrt{2 + x^2 + 5y^2} &= r \\ 2 + x^2 + 5y^2 &= r^2 \\ x^2 + 5y^2 &= r^2 - 2 \end{aligned}$$

We do not need to solve this equation—all we need is one value for x and one value for y. So, take $y = 0$. From $x^2 = r^2 - 2$ we get $x = \pm\sqrt{r^2 - 2}$. To check:

$$\begin{aligned} g(x, y) &= g(\pm\sqrt{r^2 - 2}, 0) \\ &= \sqrt{2 + (\pm\sqrt{r^2 - 2})^2 + 5(0)^2} \\ &= \sqrt{r^2} \\ &= |r| = r \end{aligned}$$

(since $r \in [\sqrt{2}, \infty)$, r is positive, and thus $|r| = r$).

23. The domain of g is $\mathbb{R}^2$, since the absolute value function is defined for all real numbers.

The range of g is $[0, \infty)$: pick a real number $r \in [0, \infty)$; Then $g(0, r) = 3|0| + |r| = |r| = r$ (the last equality is true because $r \geq 0$).

(Of course, there are other choices for x and y; for instance, $g(r/3, 0) = 3|r/3| + |0| = |r| = r$ or

$$g(r/6, r/2) = 3|r/6| + |r/2| = |r/2| + |r/2| = |r| = r$$

and so on.)

25. Since $\arctan t$ is defined for all real numbers, it follows that the domain of f is $\mathbb{R}^2$. As well, $\arctan t < \pi/2$ for all $t \in \mathbb{R}$, and $\arctan t \geq 0$ if $t \geq 0$. Because $t = x^2 \geq 0$, the range of f is a subset of $[0, \pi/2)$.

We now prove that that the range is *equal* to $[0, \pi/2)$. Pick $r \in [0, \pi/2)$; from $\arctan(x^2) = r$ we get $x^2 = \tan r$ and $x = \pm\sqrt{\tan r}$. (Note that $r \in [0, \pi/2)$ guarantees that $\tan r \geq 0$, and so the square root is defined.) The function f does not depend on y. Thus (picking any value for y we wish)

$$\begin{aligned} f(\pm\sqrt{\tan r}, y) &= \arctan(\pm\sqrt{\tan r})^2 \\ &= \arctan(\tan r) \\ &= r \end{aligned}$$

Hence the range of f is all of $[0, \pi/2)$.

27. Think of $f(x, y, z) = e^{-2x^2-3y^2-z^2} = e^{-(2x^2+3y^2+z^2)}$. We analyze the exponent:

$$\begin{aligned} 2x^2 + 3y^2 + z^2 &\geq 0 \\ -(2x^2 + 3y^2 + z^2) &\leq 0 \\ e^{-(2x^2+3y^2+z^2)} &\leq e^0 = 1 \end{aligned}$$

Since $e^t > 0$ for all $t \in \mathbb{R}$, we conclude that the range of f is contained in $(0, 1]$.

Pick $r \in (0, 1]$; then

$$\begin{aligned} f(x, y, z) = e^{-2x^2-3y^2-z^2} &= r \\ -2x^2 - 3y^2 - z^2 &= \ln r \end{aligned}$$

Take $x = y = 0$; then $-z^2 = \ln r$, $z^2 = -\ln r$, and $z = \pm\sqrt{-\ln r}$. (Note that, because $r \in (0, 1]$, $\ln r \leq 0$, and thus the square root is defined.) We have just shown that $f(0, 0, \pm\sqrt{-\ln r}) = r$, which completes the proof that the range of f is *equal* to the interval $(0, 1]$.

29. The distance between a point (x, y, z) and $(3, 2, -4)$ is given by the function

$$d(x, y, z) = \sqrt{(x-3)^2 + (y-2)^2 + (z+4)^2}$$

It is given that f is proportional to d; thus, $f = kd$, i.e.,

$$f(x, y, z) = k\sqrt{(x-3)^2 + (y-2)^2 + (z+4)^2}$$

where k is a constant. Using $f(1, 1, 3) = 18$ we find the value of k:

$$\begin{aligned} 18 &= k\sqrt{(1-3)^2 + (1-2)^2 + (3+4)^2} \\ 18 &= k\sqrt{54} = k\sqrt{9 \cdot 6} = 3k\sqrt{6} \\ k &= \frac{18}{3\sqrt{6}} = \frac{6}{\sqrt{6}} = \sqrt{6} \end{aligned}$$

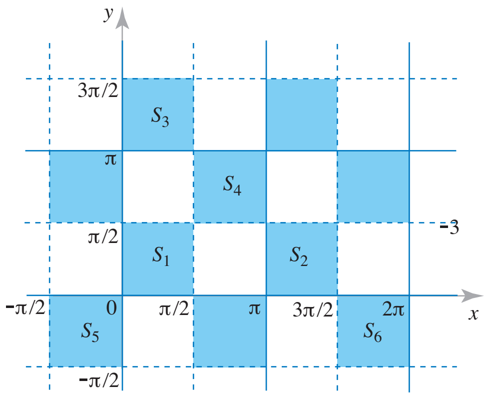
Thus, $f(x, y, z) = \sqrt{6}\sqrt{(x-3)^2 + (y-2)^2 + (z+4)^2}$.

31. A point (x, y) is in the domain of f if $\tan x \tan y \geq 0$. Thus, (a) $\tan x \geq 0$ and $\tan y \geq 0$, or (b) $\tan x \leq 0$ and $\tan y \leq 0$.

Recall that $\tan t \geq 0$ whenever t is in $[0 + k\pi, \pi/2 + k\pi)$ where k is an integer, and $\tan t \leq 0$ if $t \in (-\pi/2 + k\pi, 0 + k\pi]$.

From (a) we conclude that $x \in [0 + k_1\pi, \pi/2 + k_1\pi)$ and $y \in [0 + k_2\pi, \pi/2 + k_2\pi)$ for integers k_1 and k_2. Taking $k_1 = k_2 = 0$, we obtain $x \in [0, \pi/2)$ and $y \in [0, \pi/2)$, which represents the square S_1 in the figure below. By keeping $k_2 = 0$ and varying k_1, we obtain the remaining squares in the same row. The square S_2 is represented by $k_1 = 1$ and $k_2 = 0$, i.e., $x \in [\pi, 3\pi/2)$ and $y \in [0, \pi/2)$. By taking non-zero values for k_2, and varying k_1, we obtain the squares in every other row. For instance, to obtain S_3, we take $k_1 = 0$ and $k_2 = 1$, i.e., $x \in [0, \pi/2)$ and $y \in [\pi, 3\pi/2)$.

The remaining squares are obtained from (b): $x \in (-\pi/2 + k_3\pi, 0 + k_3\pi]$ and $y \in (-\pi/2 + k_4\pi, 0 + k_4\pi]$, for integers k_3 and k_4. For instance, S_4 is represented by $k_3 = k_4 = 1$, S_5 is represented by $k_3 = k_4 = 0$, and S_6 is represented by $k_3 = 2$ and $k_4 = 0$.



33. From $\text{BMI}(m,h) = m/h^2$ we get

$$\begin{aligned}\text{BMI}(m+0.1m, h+0.1h) = \text{BMI}(1.1m, 1.1h) &= \frac{1.1m}{(1.1h)^2} \\ &= \frac{1}{1.1}\frac{m}{h^2} \\ &\approx 0.91\text{BMI}(m,h)\end{aligned}$$

Thus, the body mass index of a person A who is 10% heavier and 10% taller than a person B is about 91% of the body mass index of B.

35. Using $S_D(m,h) = 0.20247m^{0.425}h^{0.725}$ we calculate

$$\begin{aligned}S_D(1.1m, 1.05h) &= 0.20247(1.1m)^{0.425}(1.05h)^{0.725} \\ &= 0.20247(1.1)^{0.425}m^{0.425}(1.05)^{0.725}h^{0.725} \\ &= (1.1)^{0.425}(1.05)^{0.725}0.20247m^{0.425}h^{0.725} \\ &\approx 1.07883S_D(m,h)\end{aligned}$$

and

$$\begin{aligned}S_D(1.05m, 1.1h) &= 0.20247(1.05m)^{0.425}(1.1h)^{0.725} \\ &= 0.20247(1.05)^{0.425}m^{0.425}(1.1)^{0.725}h^{0.725} \\ &= (1.05)^{0.425}(1.1)^{0.725}0.20247m^{0.425}h^{0.725} \\ &\approx 1.09399S_D(m,h)\end{aligned}$$

Thus, of the two options: a 10% increase in mass and a 5% increase in height, or a 5% increase in mass and a 10% increase in height, the latter makes the body surface area larger.

Section 2 Graph of a Function of Several Variables

1. A level curve of a function f is a set of points in the domain of f at which f takes on the same value.

Level curves corresponding to different values cannot intersect. (Think of a topographic map: is it possible that the contour curves representing different altitudes intersect?) Here is a formal proof: assume that the level curves $f(x,y)=c_1$ and $f(x,y)=c_2$ ($c_1 \neq c_2$) have a point (x_0,y_0) in common. As this point, $f(x_0,y_0)=c_1$ and $f(x_0,y_0)=c_2$, which violates the definition of a function as a unique assignment.

3. The contour diagram of a linear function consists of (i) parallel lines that are (ii) equally spaced. The diagram (a) does not satisfy (ii), and (b) does not satisfy (i). Since both (i) and (ii) are true for (c), we conclude that (c) is the only diagram (of the three offered) which represents a linear function.

5. One trick is to use a function of two variables that depends on one variable only, such as $f(x,y)=x^3$. The level curves of f are given by $f(x,y)=x^3=c$, i.e., $x=\sqrt[3]{c}$ (this equation represents a family of vertical lines, which are not equally spaced). Using the same idea: the contour diagrams of $g(x,y)=e^y$, $h(x,y)=1/(x+4)$, and $k(x,y)=\sin y$ consist of parallel lines, either horizontal or vertical.

An alternative is to compose a linear function of two variables with a one-variable function, such as $f(x,y)=e^{3x+y-4}$. The level curves of f are given by

$$f(x,y)=e^{3x+y-4}=c$$
$$3x+y-4=\ln c$$
$$y=-3x+4+\ln c$$

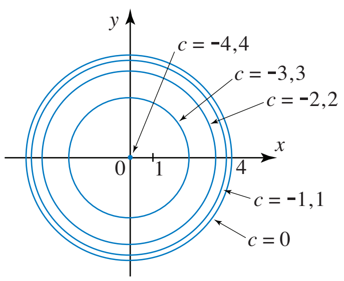
Thus, the level curves are lines of slope -3. Few more examples: $g(x,y)=(x-4y+3)^3$, $h(x,y)=1/(x-y+2)$, and $k(x,y)=\ln(3+x+6y)$.

7. Starting with $g(x,y)=\sqrt{16-x^2-y^2}=c$, we get

$$16-x^2-y^2=c^2$$
$$x^2+y^2=16-c^2$$

Since $x^2+y^2 \geq 0$ for all x and y, it follows that $16-c^2 \geq 0$, i.e., $c^2 \leq 16$ and $-4 \leq c \leq 4$.

We conclude that: (i) there are no level curves if $c<-4$ or $c>4$; (ii) if $c=\pm 4$, the level "curve" $x^2+y^2=0$ consists of a single point, $(0,0)$; (iii) if $-4<c<4$, the level curve of value c is a circle of radius $\sqrt{16-c^2}$ centred at the origin.



9. From $f(x,y)=x-4=c$ we get $x=c+4$; i.e., the level curves of f are vertical lines. The level curve of value c intersects the x-axis at $x=c+4$.

11. From $f(x,y)=ye^x=c$ we get $y=ce^{-x}$, and it follows that the level curves of f are transformed versions of the graph of e^{-x}. The level curve $y=ce^{-x}$ is obtained by vertically expanding the graph of e^{-x} by a factor of c if $c>1$ and by vertically contracting it if $0<c<1$. If $c=0$, the level curve $y=ce^{-x}=0$ coincides with the x-axis. For the negative values of c, we need to reflect: If $c<-1$, the graph of $y=ce^{-x}$ is obtained by vertically expanding the graph of e^{-x} by a factor of $|c|$ and then reflecting it across the x-axis. If $-1 < c < 0$, we vertically contract the graph of e^{-x} by a factor of $|c|$ and then reflect it across the x-axis.

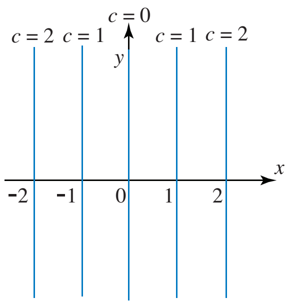
13. From $f(x,y) = y - \cos x = c$ we get $y = \cos x + c$. Thus, a level curve of value c is a vertical shift (up if $c > 0$ and down if $c < 0$) of the graph of $y = \cos x$. The level curve of value $c = 0$ is the curve $y = \cos x$.

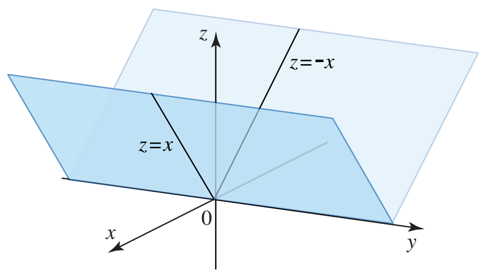
15. From $f(x,y) = \sqrt[3]{xy} = c$ we get $xy = c^3$ and $y = c^3/x$. If $c \neq 0$, the level curve of f is a scaled graph of $y = 1/x$ if $c > 0$ and scaled and reflected (across the x-axis) graph of $y = 1/x$ if $c < 0$. By "scaled" we mean: expanded vertically by a factor of $|c|$ if $c > 1$ or $c < -1$ and compressed vertically by a factor of $|c|$ if $-1 < c < 1$.

If $c = 0$ then $xy = c^3$ implies that $x = 0$ or $y = 0$; the level curve of value $c = 0$ consists of the x-axis and the y-axis.

17. From $g(x,y) = |x| = c$ we conclude that there are no level curves corresponding to negative values of c. If $c = 0$, then $|x| = 0$ implies that $x = 0$; thus, the level curve of value $c = 0$ is the y-axis. When $c > 0$, $|x| = c$ implies $x = \pm c$. This means that the level curve of value c consists of a pair of vertical lines $x = \pm c$ (see the figure below).

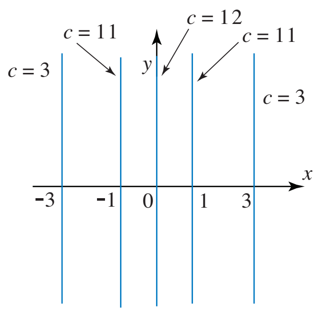
The surface $g(x,y) = |x|$ intersects the xz-plane along $z = |x|$. Since the function g does not depend on y, the intersection of the graph of g with any plane parallel to the xz-plane is a copy of $z = |x|$. Thus, the graph of g is obtained by moving the graph of $z = |x|$ along the y-axis; see the figure below.

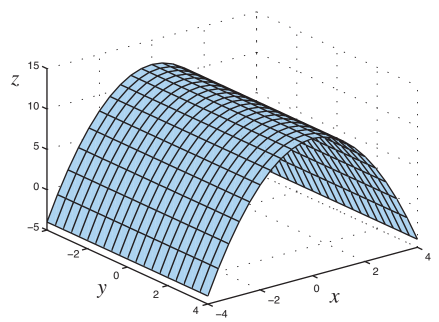




19. Note that $f(x,y) = 12 - x^2 \leq 12$. Thus, there are no level curves of value larger than 12. Assuming that $c \leq 12$, from $f(x,y) = 12 - x^2 = c$ we get $x^2 = 12 - c$ and $x = \pm\sqrt{12-c}$. So the level curve of value $c = 12$ is $x = 0$, i.e., the y-axis. If $c < 12$, the level curve of value c consists of a pair of vertical lines $x = \pm\sqrt{12-c}$; see the figure below.

The surface $f(x,y) = 12 - x^2$ intersects the xz-plane along $z = 12 - x^2$. Since the function f does not depend on y, the intersection of the graph of f with any plane parallel to the xz-plane is a copy of $z = 12 - x^2$. Thus, the graph of f is obtained by moving the graph of $z = 12 - x^2$ along the y-axis; see the figure below.

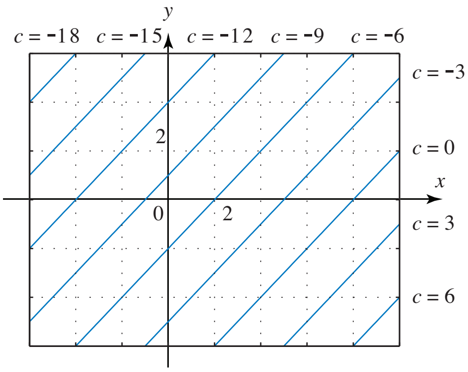


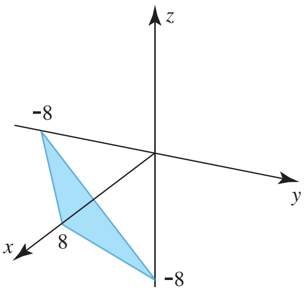


21. From $f(x,y) = x - y - 8 = c$ we get $y = x - 8 - c$. The level curve of value c is a line of slope

1 whose y-intercept is $-8-c$. The contour diagram consists of equally spaced parallel lines (f is a linear function); see below.

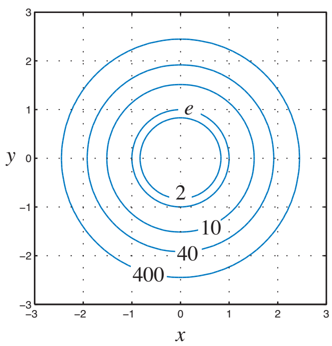
The plane cuts the yz-plane along the line $z=-y-8$, and the xz-plane along the line $z=x-8$. The x-intercept (substitute $y=0$ and $z=0$ into the formula for the function) is $x=8$. The y-intercept ($x=z=0$) is $y=-8$, and the z-intercept is -8.

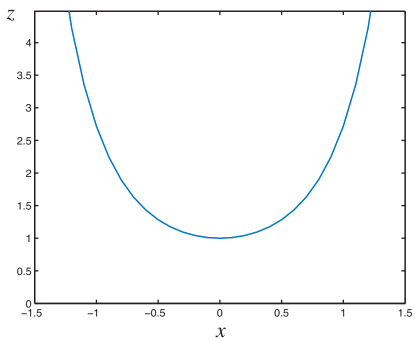


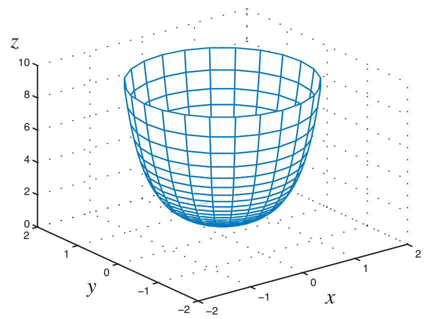


23. Since $x^2+y^2\geq 0$, it follows that $e^{x^2+y^2}\geq e^0=1$. Thus, f has level curves of value 1 or larger than 1 only. From $f(x,y)=e^{x^2+y^2}=c$ we get $x^2+y^2=\ln c$. Therefore: if $c<1$, there are no level curves; if $c=1$, the level curve $x^2+y^2=\ln 1=0$ is the point $(0,0)$; if $c>1$, the level curve is a circle of radius $\sqrt{\ln c}$ centred at the origin. The figure below (left) shows several level curves.

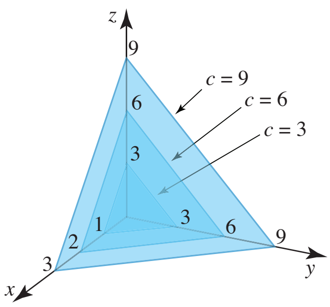
The fact that the level curves are circles implies that the graph of f is rotationally symmetric (i.e., symmetric with respect to the z-axis). The intersection of the graph of f with the xz-plane is the curve $z=e^{x^2+0^2}=e^{x^2}$; see the figure below (centre). The graph of f is obtained by rotating this curve about the z-axis (see figure below, right).

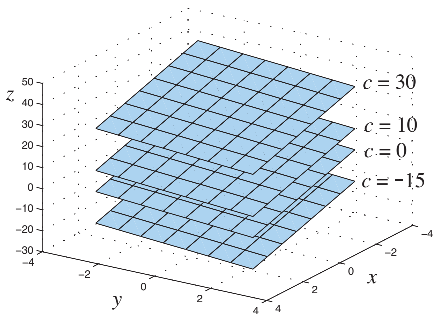






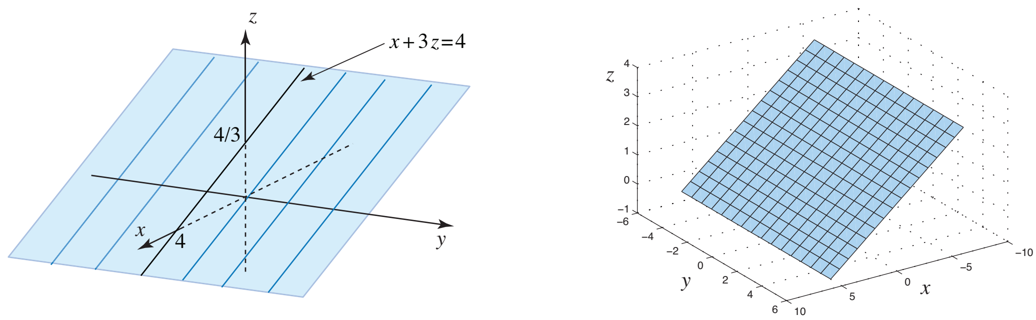
25. The level surfaces of f, given by $f(x,y,z)=3x+y+z=c$, where $c\in\mathbb{R}$, are parallel planes in $\mathbb{R}^3$. The plane $3x+y+z=c$ intersects the x-axis ($y=z=0$) at $x=c/3$, the y-axis ($x=z=0$) at $y=c$, and the z-axis ($x=y=0$) at $z=c$. In the figure below we show the parts of the three planes (level surfaces of values 3, 6, and 9) in the first octant, and a computer-generated picture of several planes $3x+y+z=c$.





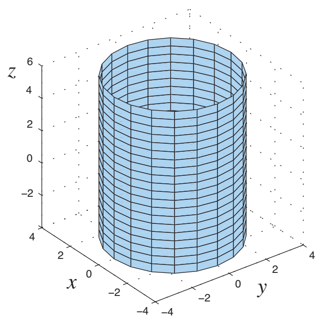
27. The fact that there is no y in the equation makes is easier: we draw the graph of $x+3z=4$ in the xz-plane, and then move it along the y-axis.

The equation $x+3z=4$ represents the line whose x-intercept is 4 and z-intercept $4/3$. By moving this line parallel to itself along the y-axis, we generate the graph of the equation $x+3z=4$ in space. The figure below shows two different views of the plane.



29. In the xy-plane, the equation $x^2+y^2-10=0$ represents the circle of radius $\sqrt{10}$ centred at the origin. Since there is no z, we conclude that the intersection of the surface $x^2+y^2-10=0$ with any horizontal plane is the same circle.

In other words, the surface $x^2+y^2-10=0$ is generated by moving the circle $x^2+y^2-10=0$ vertically up and down, keeping its centre on the z-axis. The surface thus obtained is a cylinder of radius $\sqrt{10}$, whose axis of symmetry is the z-axis. See the figure below.



31. The distance between a fixed point (a,b,c) and a point (x,y,z) in $\mathbb{R}^3$ is given by

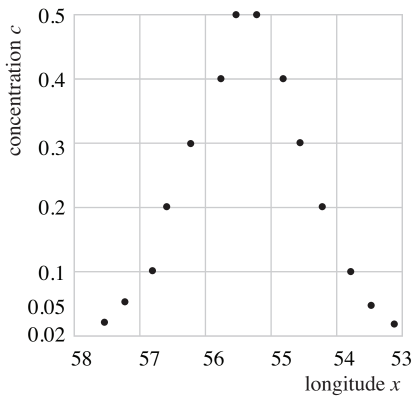
$$d=\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$$

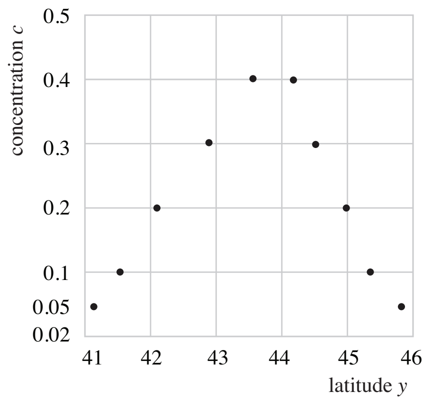
All points (x,y,z) whose distance d from (a,b,c) is equal to r lie on a sphere of radius r centred at the origin. Squaring the equation $d=r$, we get

$$d^2=(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

33. As we walk along the horizontal line representing the latitude of 44°, we record the values of the level curves that we meet, and the location where we meet them. Going from left to right, we meet the level curves of value 0.02 and 0.05 before we arrive at 57° mark. Between 57° and 56°, we cross the level curves of values 0.1, 0.2 and 0.3 at about equal distances from each other. By continuing to mark the values of the level curves and the corresponding locations where we cross them, we obtain the points in the diagram below (left). Once done, we can connect the points with a smooth curve.

The diagram on the right is obtained in the same way, this time by walking vertically along the line representing the longitude of 55°.

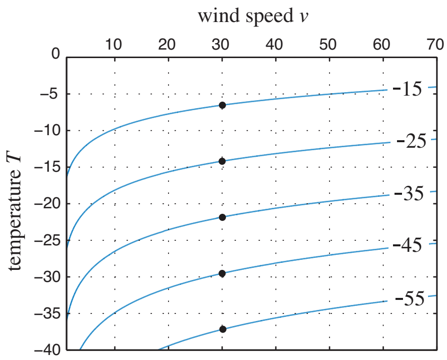




35. The five points are shown in the figure below. They are equally spaced, for the following reason. Substituting $v = 30$ into $W(T, v) = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$ we obtain

$$W(T, 30) = 13.12 + 0.6215T - 11.37(30)^{0.16} + 0.3965T(30)^{0.16} = -6.4730 + 1.3048T$$

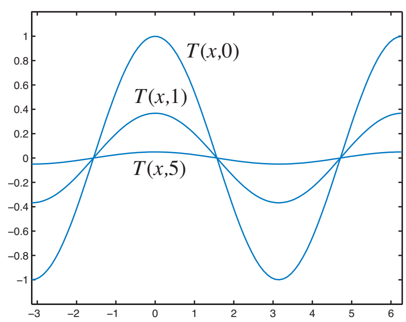
which is a linear function. Thus, the same difference in T (say, moving from -25 to -35 and moving from -35 to -45) will generate the same difference in W.

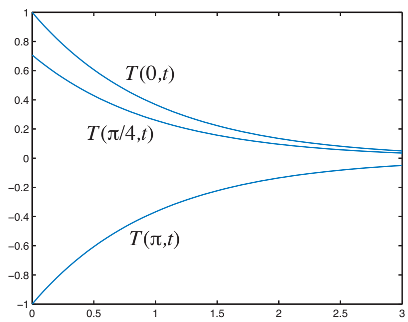


37. The graphs of $T(x, t_0)$ for $t_0 = 0$, $t_0 = 1$, and $t_0 = 5$ are shown in the figure below, left. The three curves represent the snapshots of the temperature initially, 1, and 5 time units later. At the locations where it is above zero, the temperature decreases with time, and at the locations where it's below zero, the temperature increases with time. There is no change in temperature at all locations where initially the temperature is zero.

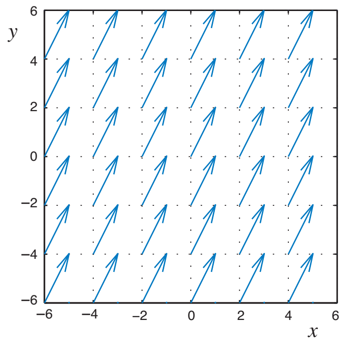
The graphs of the curves $T(x_0, t)$ for $x_0 = 0$, $x_0 = \pi/4$, and $x_0 = \pi$ are shown below, right. The three curves describe how the temperature changes at three fixed locations. At the locations $x_0 = 0$ and $x_0 = \pi/4$, the temperature is initially positive and then decreases exponentially. At the location $x_0 = \pi$ the temperature is initially negative, and then increases. All three graphs approach 0 as $t \to \infty$.

To put it all together: initially the temperature is distributed according to $T(x, 0) = \cos x$. Over time, the temperature evens out, approaching the value of 0 at all locations x.

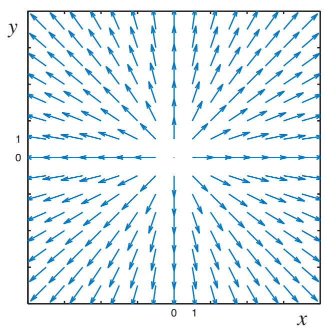




39. The vector field $\mathbf{F}(x, y) = \mathbf{i} + 2\mathbf{j}$ is a constant vector field; it is represented by the same vector $\mathbf{i} + 2\mathbf{j}$ starting at every point in its domain $\mathbb{R}^2$.



41. The vector $x\mathbf{i} + y\mathbf{j}$ is the position vector of a point (x, y); thus, $x\mathbf{i} + y\mathbf{j}$ points radially away from the origin (i.e., from the origin towards (x, y)). The square root in the denominator is the length of $x\mathbf{i} + y\mathbf{j}$, and so $\mathbf{F}(x, y) = (x\mathbf{i} + y\mathbf{j})/\sqrt{x^2 + y^2}$ is a unit vector field. Its direction is radial, away from the origin; see the figure below.



43. The vector field $\mathbf{F} = y\mathbf{i}$ has no $\mathbf{j}$ component, and so its direction is parallel to the x-axis. Thus, the diagram (c) represents $\mathbf{F}$. Writing $\mathbf{G} = -x\mathbf{i} - y\mathbf{j} = -(x\mathbf{i} + y\mathbf{j})$, we see that the direction of G is radial toward the origin (i.e., opposite of the direction of the position vector of a point (x, y)). Thus, (b) represents $\mathbf{G}$. The components $\mathbf{H} = \sin x\mathbf{i} + \sin y\mathbf{j}$ are periodic functions, and we see the periodicity in the diagram (a).

Section 3 Limits and Continuity

1. The fact that $\lim_{(x,y)\to(1,4)} f(x,y) = 5$ means that we can make the values of f as close to 5 as desired by picking points (x,y) close enough to $(1,4)$. More precisely: suppose we wish to make the values of f to fall within 0.01 of 5, i.e., to fall inside the interval $(4.99, 5.01)$. We can find an open disk centred at $(1,4)$ such that for every point (x,y) in that disk (except possibly for $(1,4)$) the value of the function $f(x,y)$ is in the interval $(4.99, 5.01)$. The fact that the limit is 5 means that we can repeat this procedure for any real number (i.e., we can replace 0.01 by any real number), no matter how small.

3. To prove that a function $f(x,y)$ is continuous at the point $(2,-1)$, we have to verify that $\lim_{(x,y)\to(2,-1)} f(x,y) = f(2,-1)$. Since

$$f(2,-1) = \frac{(2)^3(-1) - (2)(-1)^2}{(2)^2 + (-1)^2} = \frac{-10}{5} = -2$$

this amounts to showing that

$$\lim_{(x,y)\to(2,-1)} \frac{x^3y - xy^2}{x^2 + y^2} = -2$$

5. The requirement for continuity states that

$$\lim_{(x,y)\to(0,0)} \frac{\cos(xy)}{x^2 + y^2 + 1} = f(0,0) = 0.$$

To figure out the limit, consider the approach along the x-axis (i.e., when $y = 0$):

$$\lim_{(x,y)\to(0,0)} \frac{\cos(xy)}{x^2 + y^2 + 1} = \lim_{x\to 0} \frac{\cos(0)}{x^2 + 1} = \frac{1}{1} = 1$$

Thus, if the limit of $f(x,y)$ as (x,y) approaches $(0,0)$ exists, then it must be equal to 1. Or else, the limit does not exist. In either case, the limit is not zero, and hence f is not continuous at $(0,0)$.

7. Approaching $(0,0)$ along lines does not amount to checking *all* possible paths. Thus, all we can say is that *if* the limit exists, then it must be equal to 3. (However, as we said, the limit might not exist.)

9. The given function is a polynomial; thus, using the direct substitution,

$$\lim_{(x,y)\to(2,0)} (xy - 2x^2) = (2)(0) - 2(2)^2 = -8$$

11. The given function involves a polynomial and a rational function. By direct substitution,

$$\lim_{(x,y)\to(1,-3)} \left(x - \frac{xy}{2y - 11}\right) = \left(1 - \frac{(1)(-3)}{2(-3) - 11}\right) = 1 - \frac{3}{17} = \frac{14}{17}$$

13. The given function is a rational function. By direct substitution,

$$\lim_{(x,y)\to(0,-2)} \frac{x^2 - y^2}{x^2 + y^2} = \frac{(0)^2 - (-2)^2}{(0)^2 + (-2)^2} = \frac{-4}{4} = -1$$

15. Along the line $y = 2x$,

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{2x^2 - 3y^2} = \lim_{x\to 0} \frac{2x(2x)}{2x^2 - 3(2x)^2} = \lim_{x\to 0} \frac{4x^2}{-10x^2} = \lim_{x\to 0} \frac{4}{-10} = -\frac{2}{5}$$

Along the line $y = 7x$,

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{2x^2 - 3y^2} = \lim_{x\to 0} \frac{2x(7x)}{2x^2 - 3(7x)^2} = \lim_{x\to 0} \frac{14x^2}{-145x^2} = \lim_{x\to 0} \frac{4}{-145} = -\frac{4}{145}$$

Since the two paths to $(0,0)$ give different values, we conclude that the limit does not exist.

17. Let's choose simplest paths. Along $y = 0$,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{(x^2+y^2)^{3/2}} = \lim_{x\to 0} \frac{0}{(x^2)^{3/2}} = \lim_{x\to 0} 0 = 0$$

The limit along $x = 0$ is 0 as well, since the function is symmetric in x and y. Along $y = x$,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{(x^2+y^2)^{3/2}} = \lim_{x\to 0} \frac{x^2}{(x^2+x^2)^{3/2}} = \lim_{x\to 0} \frac{x^2}{(2x^2)^{3/2}} = \lim_{x\to 0} \frac{x^2}{2^{3/2}x^3} = \frac{1}{2^{3/2}} \lim_{x\to 0} \frac{1}{x}$$

This limit is not a real number, and we are done: we found two paths ($y = 0$ and $y = x$) along which the values for the limit differ.

19. The function $g(x,y) = e^{-x-y-2}$ is continuous for all (x,y), as the composition of the exponential function $h(t) = e^t$ (which is continuous for all real numbers t) and the linear function (polynomial) $l(x,y) = -x - y - 2$ (which is continuous for all $(x,y) \in \mathbb{R}^2$). The constant function 1 is continuous, and thus the difference $f(x,y) = 1 - e^{-x-y-2}$ is continuous for all $(x,y) \in \mathbb{R}^2$. We conclude that

$$\lim_{(x,y)\to(0,0)} \left(1 - e^{-x-y-2}\right) = \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 1 - e^{-2}$$

21. The function $h(x,y) = e^{-xy}$ is the composition of the exponential function e^t (continuous at all t) and the polynomial in two variables $-xy$ (continuous at all (x,y)), and therefore continuous at all $(x,y) \in \mathbb{R}^2$. The function $f(x,y) = e^{-xy} - x + 4$ is continuous at all (x,y), as the sum of two continuous functions ($l(x,y) = -x + 4$ is a polynomial, hence continuous). Since $f(1,10) = e^{-10} - 1 + 4 = e^{-10} + 3 > 0$, the function $F(x,y) = \ln(e^{-xy} - x + 4)$ is continuous at $(1,10)$. Thus,

$$\lim_{(x,y)\to(1,10)} \ln\left(e^{-xy} - x + 4\right) = F(1,10) = \ln\left(e^{-10} - 1 + 4\right) = \ln(e^{-10} + 3) \approx 1.09862$$

23. The function $h(t) = \sqrt{t}$ is continuous as long as $t \geq 0$. The function $g(x,y) = 1 + x^2 + y^2$ is a polynomial, and hence continuous at all (x,y). Because $g(x,y) = 1 + x^2 + y^2 \geq 1 \geq 0$ for all (x,y), the composition

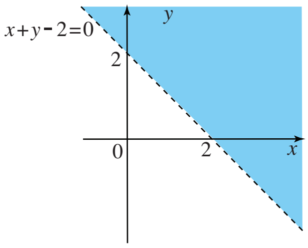
$$f(x,y) = (h \circ g)(x,y) = h(1 + x^2 + y^2) = \sqrt{1 + x^2 + y^2}$$

is defined and continuous at all $(x,y) \in \mathbb{R}^2$.

25. The function $f(m,h) = m/h^2$ is continuous at all (m,h) such that $h \neq 0$. In terms of the body mass index $h = 0$ makes no sense, so BMI is continuous on any biologically meaningful subset of the values for m and h. One possible domain is the set of all (m,h), where m ranges from the mass of the lightest to the mass of the heaviest person on Earth, and h ranges from the height of the shortest to the height of the tallest person on Earth.

27. The function $f(t) = \ln t$ is defined and continuous for $t > 0$. Therefore, $g(x,y) = \ln(x + y - 2)$ is continuous at all points (x,y) which satisfy $x + y - 2 > 0$.

The line $x + y - 2 = 0$ divides the xy-plane into two half-planes. To figure out which half-plane is the domain, we use a test point, for instance $(0,0)$. Since $0 + 0 - 2 > 0$ does not hold, the region defined by $x + y - 2 > 0$ is the one that does not contain the origin. Thus, the function g is continuous on the shaded region in the figure below. (Recall that dashed line indicates that the points on the line are not included in the domain.)



29. From $e^x - 1 = 0$ we get $e^x = 1$ and $x = 0$. Thus, f is continuous at all points (x, y) in $\mathbb{R}^2$ such that $x \neq 0$. Geometrically, the largest domain where f is continuous is the xy-plane with the y-axis removed.

31. We investigate the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ by computing its values along different paths that lead to $(0, 0)$. Along $x = 0$,

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + 2y^2} = \lim_{y\to 0} \frac{0}{y^2} = \lim_{y\to 0} 0 = 0$$

This does not help, since we did not prove that $\lim_{(x,y)\to(0,0)} f(x, y) = 0$. The limit along $y = 0$ is 0 as well. Let's try $y = x$:

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + 2y^2} = \lim_{x\to 0} \frac{2x^2}{3x^2} = \lim_{x\to 0} \frac{2}{3} = \frac{2}{3}$$

Thus $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist, and we conclude that f is not continuous at $(0, 0)$.

33. Assume that $m \neq 0$. Along $y = mx$,

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2} = \lim_{x\to 0} \frac{x^3(mx)}{x^6 + (mx)^2} = \lim_{x\to 0} \frac{mx^4}{x^6 + m^2x^2} = \lim_{x\to 0} \frac{mx^2}{x^4 + m^2} = \frac{0}{m^2} = 0$$

Along $y = 0$ (i.e., when $m = 0$),

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2} = \lim_{x\to 0} \frac{0}{x^6 + 0} = \lim_{x\to 0} 0 = 0$$

Along $x = 0$,

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2} = \lim_{y\to 0} \frac{0}{0 + y^2} = \lim_{y\to 0} 0 = 0$$

Thus, indeed, along all lines through the origin, the limit is zero. Along $y = x^3$,

$$\lim_{(x,y)\to(0,0)} \frac{x^3 x^3}{x^6 + (x^3)^2} = \lim_{x\to 0} \frac{x^6}{x^6 + x^6} = \lim_{x\to 0} \frac{x^6}{2x^6} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

We conclude that the limit in question does not exist.

35. Along the y-axis ($x = 0$) the limit is

$$\lim_{(x,y)\to(0,0)} \arctan(1.2x/y) = \lim_{y\to 0} \arctan(0/y) = \arctan 0 = 0$$

Along the line $y = 1.2x$,

$$\lim_{(x,y)\to(0,0)} \arctan(1.2x/y) = \lim_{x\to 0} \arctan(1.2x/1.2x) = \arctan 1 = \pi/4$$

Thus, the limit does not exist.

Section 4 Partial Derivatives

1. The change is with respect to the variable t, and thus

$$f_t(x, y, t) = \lim_{h\to 0} \frac{f(x, y, t+h) - f(x, y, t)}{h}$$

3. We see that $f(2, 3) = 4$. Moving vertically up, we meet the level curve of value 5 (approximately) at $(2, 4.5)$; i.e., $f(2, 4.5) = 5$. Thus

$$\frac{\partial f}{\partial y}(2, 3) \approx \frac{f(2, 4.5) - f(2, 3)}{4.5 - 3} = \frac{5 - 4}{4.5 - 3} = \frac{1}{1.5} = \frac{2}{3}$$

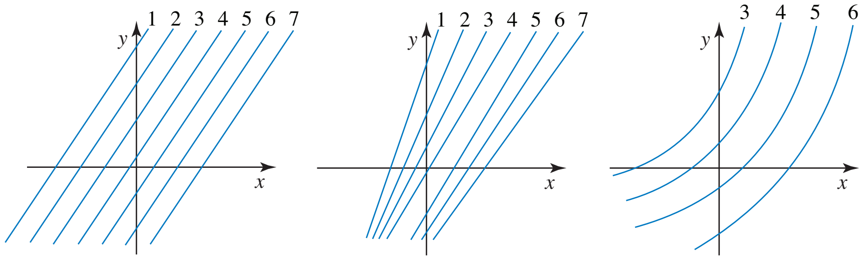
Using the fact that $f(2, 2) = 3$, we get another estimate:

$$\frac{\partial f}{\partial y}(2, 3) \approx \frac{f(2, 2) - f(2, 3)}{2 - 3} = \frac{3 - 4}{2 - 3} = 1$$

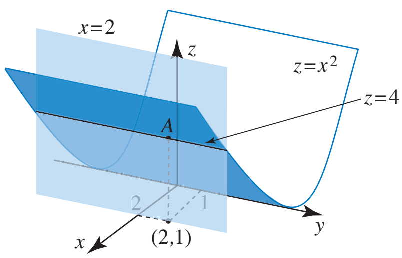
Taking the average, we obtain $\partial f/\partial y(2, 3) \approx (2/3 + 1)/2 = 5/6$.

5. Here is an idea: the partial derivatives of a linear function $f(x, y) = ax + by + c$ are $\partial f/\partial x = a$ and $\partial f/\partial y = b$. Since we need to satisfy $\partial f/\partial x > 0$ and $\partial f/\partial y < 0$ we pick a positive value for a and a negative value for b, say $f(x, y) = 2x - y + 3$. The level curves are given by $f(x, y) = 2x - y + 3 = c$, i.e., $y = 2x + 3 - c$. Thus, we draw parallel lines with the slope of 2 (see below, left).

We are not asked to draw level curves of a linear function, so we can expand the idea: the level curves need not be parallel, nor equally spaced; as well, we can use other curves; see the figures below. To check: in the horizontal direction away from any point the values of f increase, and in the vertical direction the values decrease.



7. The graph of the surface $z = x^2$ is the parabolic sheet shown in the figure below. The intersection of this surface with the plane $x = 2$ is the curve $z = 2^2 = 4$ (horizontal line). The partial derivative $f_y(2, 1)$ is the slope of this line at the point (labeled A) where $y = 1$. Since the line is horizontal, the slope is zero. Thus, $f_y(2, 1) = 0$.



9. Moving away from A in the positive x-direction, the graph of f increases. The same is true if we move away from A in the positive y-direction. Thus, $f_x(A) > 0$ and $f_y(A) > 0$. For the same reason $f_x(B) > 0$ and $f_y(B) > 0$.

11. From $f_x(x,y) = 2xy^2$, we get $f_x(-2,0) = 2(-2)(0)^2 = 0$. Consider the curve which is obtained as the intersection of the graph of $f(x,y) = 3 + x^2y^2$ and the vertical plane $y = 0$ (its equation is $z = 3$, so it is a line). The slope of this line at the point whose x-coordinate is -2 is zero.

13. Keeping y constant, we obtain $f_x(x,y) = 3x^2y^3 - 7(1)y + 0 = 3x^2y^3 - 7y$. Keeping x constant, we get $f_y(x,y) = x^3 3y^2 - 7x(1) + 0 = 3x^3y^2 - 7x$.

15. Keeping t fixed, $h_x(x,t) = \cos(3x - 2t) \cdot 3 = 3\cos(3x - 2t)$. Keeping x fixed and differentiating with respect to t, we get $h_t(x,t) = \cos(3x - 2t) \cdot (-2) = -2\cos(3x - 2t)$.

17. Thinking of the exponent of e as $-x^2/t = (-1/t)x^2$ and using the chain rule, we get

$$f_x(x,t) = \frac{1}{4t}e^{-x^2/t} \cdot \left(-\frac{1}{t}\right) 2x = -\frac{x}{2t^2}e^{-x^2/t}$$

Using the product and the chain rules (this time thinking of the exponent of e as $-x^2/t = -x^2t^{-1}$,

$$\begin{aligned} f_t(x,t) &= \frac{1}{4}(-1)t^{-2}e^{-x^2/t} + \frac{1}{4t}e^{-x^2/t} \cdot (-x^2)(-1)t^{-2} \\ &= -\frac{1}{4t^2}e^{-x^2/t} + \frac{x^2}{4t^3}e^{-x^2/t} \\ &= \frac{1}{4t^2}\left(-1 + \frac{x^2}{t}\right)e^{-x^2/t} \end{aligned}$$

19. Keeping w constant and using the chain rule,

$$f_x(x,w) = w^{-3}\,\frac{1}{x + w^2} \cdot 1 = \frac{1}{w^3(x + w^2)}$$

Keeping x fixed and using the product and the chain rules,

$$f_w(x,w) = -3w^{-4}\ln(x + w^2) + w^{-3}\,\frac{1}{x + w^2} \cdot 2w = -\frac{3\ln(x + w^2)}{w^4} + \frac{2}{w^2(x + w^2)}$$

21. From

$$W(T,v) = 13.12 + 0.6215T - 11.37v^{0.16} + 0.3965Tv^{0.16}$$

we compute

$$W_T(T,v) = 0.6215 + 0.3965v^{0.16}$$

Thus, $W_T(-15,30) = 0.6215 + 0.3965(30)^{0.16} \approx 1.3048$.

23. Recall that $(\arctan t)' = 1/(1 + t^2)$. Using the chain rule,

$$f_y(x,y) = \frac{1}{1 + (3x/y)^2} \cdot 3x(-1)y^{-2} = \frac{1}{1 + \frac{9x^2}{y^2}}\,\frac{-3x}{y^2} = \frac{-3x}{y^2 + 9x^2}$$

Thus,

$$f_y(1,3) = \frac{-3(1)}{(3)^2 + 9(1)^2} = -\frac{3}{18} = -\frac{1}{6}$$

25. Writing $S(m,h) = \sqrt{mh}/6 = (\sqrt{m}/6)h^{1/2}$, we compute

$$S_h(m,h) = \frac{\sqrt{m}}{6}\,\frac{1}{2}h^{-1/2} = \frac{\sqrt{m}}{12\sqrt{h}}$$

Thus,

$$S_h(70, 1.6) = \frac{\sqrt{70}}{12\sqrt{1.6}} \approx 0.5512$$

27. The point $(6, 10)$ lies on the contour curve of value 11, so $f(6, 10) = 11$. Moving away from $(6, 10)$ in the horizontal direction (thus keeping y fixed) we meet the level curve of value 10 at approximately $(6.8, 10)$; based on this,

$$f_x(6, 10) \approx \frac{f(6.8, 10) - f(6, 10)}{6.8 - 6} = \frac{10 - 11}{0.8} = -\frac{1}{0.8} = -1.25$$

Moving toward the left, we meet the level curve of value 12 at $(4, 10)$; based on this,

$$f_x(6, 10) \approx \frac{f(4, 10) - f(6, 10)}{4 - 6} = \frac{12 - 11}{-2} = -\frac{1}{2} = -0.5$$

Taking the average, we obtain $f_x(6, 10) \approx (-1.25 - 0.5)/2 = -0.875$.

29. We base our answer on the forward quotient. Moving away from $(6, 5)$ in the direction of the positive x-axis, we see that f decreases; thus, $f_x(6, 5) < 0$. Moving away from $(6, 5)$ in the direction of the positive y-axis, the function f increases, and so $f_y(6, 5) > 0$. Thus, $f_y(6, 5) > f_x(6, 5)$.

31. The diagram shows that $f(2, 15) = 10$. The function $f(x, y)$ increases as a point (x, y) moves away from $(2, 15)$ in the direction of the positive x-axis; thus, $f_x(2, 15) > 0$. As a point (x, y) moves away from $(2, 15)$ in the direction of the positive y-axis, the values of f decrease, and so $f_y(2, 15) < 0$.

33. From

$$\frac{\partial \text{BMI}}{\partial m} = \frac{1}{h^2} \quad \text{and} \quad \frac{\partial \text{BMI}}{\partial h} = m(-2)h^{-3} = -\frac{2m}{h^3}$$

we obtain

$$\frac{\partial \text{BMI}}{\partial m}(60, 1.7) = \frac{1}{(1.7)^2} \approx 0.35 \quad \text{and} \quad \frac{\partial \text{BMI}}{\partial h}(60, 1.7) = -\frac{2(60)}{(1.7)^3} \approx -24.42$$

A person of mass 60 kg and height 1.7 m has the body mass index of $\text{BMI}(60, 1.7) = 60/(1.7)^2 \approx 20.76$. At that moment, an increase of 1kg in mass (with no change in height) will increase the body mass index of that person by approximately 0.35. As well, an increase of 1 m in height (with no change in weight) will decrease the person's body mass index by about 24.42 (this is not realistic; it's better to say that, an increase of 1 cm in the person's height with no change in mass will decrease her/his body mass index by approximately 0.2442).

35. From Table 4.2 we read $H(26, 60) = 32$. Based on $H(26, 70) = 33$, we get

$$H_h(26, 60) \approx \frac{H(26, 70) - H(26, 60)}{70 - 60} = \frac{33 - 32}{10} = 0.1$$

Based on $H(26, 50) = 30$, we get

$$H_h(26, 60) \approx \frac{H(26, 50) - H(26, 60)}{50 - 60} = \frac{30 - 32}{-10} = 0.2$$

Taking the average, we obtain $H_h(26, 60) \approx (0.1 + 0.2)/2 = 0.15$.

When the temperature is 26°C and the humidity is 60 percent, the humidex is $H(26, 60) = 32$. At that moment, a unit increase in humidity (i.e., an increase by one percent), with no change in the temperature, will increase the humidex by approximately 0.15.

37. From $z = 24 - (x-3)^2 - 2(y-2)^4$ we obtain $z_x = -2(x-3)$ and $z_x(2, 1) = -2(2-3) = 2$ (that's the slope of the hill in the easterly direction). From $z_y = -8(y-2)^3$ we compute the slope in the northerly direction $z_y(2, 1) = -8(1-2)^3 = 8$. Thus, the hill is steeper in the direction of the north.

Section 5 Tangent Plane, Linearization, and Differentiability

1. Assume that the given surface is the graph of a differentiable function $x = f(x, y)$. To construct the tangent plane at a point $(a, b, f(a, b))$ on the surface, we use the fact that two intersecting lines determine a unique plane in space.

By intersecting the given surface with the vertical planes $x = a$ and $y = b$ we obtain two curves that pass through $(a, b, f(a, b))$. The tangent lines to these curves at the point $(a, b, f(a, b))$ are the lines that define the tangent plane.

3. From one-variable calculus we know that $\sin t \approx t$ if t is near 0. (This follows from the fact that the tangent line to $y = \sin t$ at $t = 0$ has the equation $y = t$.) Replacing t by $x + y$, we obtain $\sin(x + y) \approx x + y$ for (x, y) near $(0, 0)$.

Alternatively: the function $f(x, y) = \sin(x + y)$ is differentiable at all $(x, y) \in \mathbb{R}^2$, since both partial derivatives $f_x(x, y) = \cos(x + y)$ and $f_y(x, y) = \cos(x + y)$ are continuous at all (x, y). Thus, $f(x, y) \approx L_{(0,0)}(x, y)$ for (x, y) near $(0, 0)$.

From $f_x(0, 0) = \cos 0 = 1$ and $f_y(0, 0) = \cos 0 = 1$ we obtain

$$L_{(0,0)}(x, y) = f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0) = 0 + 1(x - 0) + 1(y - 0) = x + y$$

Thus, $\sin(x + y) = f(x, y) \approx L_{(0,0)}(x, y) = x + y$ for (x, y) near $(0, 0)$.

5. The contour curve of value c is given by $f(x, y) = \sqrt{x^2 + y^2} = c$, i.e., $x^2 + y^2 = c^2$ (so it is a circle of radius c, $c \geq 0$). As we keep zooming in, the contour diagram does not change, we still see circles (of smaller and smaller radii). All we can say is that this *might* mean that the function is not differentiable at $(0, 0)$, as its contour diagram does not resemble the contour diagram of a linear function.

For instance, $h(x, y) = x^2 + y^2$ is differentiable at all $(x, y) \in \mathbb{R}^2$, and, in particular, at $(0, 0)$. Its contour diagram around the origin consists of concentric circles.

7. Let $f(x, y) = \ln(x^2 - y^2)$. Note that f is defined and continuous near $(1, 0)$. The partial derivatives are:

$$f_x(x, y) = \frac{1}{x^2 - y^2}\, 2x = \frac{2x}{x^2 - y^2} \quad \text{and} \quad f_x(1, 0) = \frac{2}{1} = 2$$

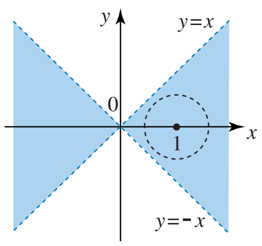
$$f_y(x, y) = \frac{1}{x^2 - y^2}\,(-2y) = -\frac{2y}{x^2 - y^2} \quad \text{and} \quad f_y(1, 0) = 0$$

Since f_x and f_y are continuous near $(1, 0)$, we conclude that f is differentiable at $(1, 0)$, and therefore $f(x, y) \approx L_{(1,0)}(x, y)$ for (x, y) near $(1, 0)$. We compute $f(1, 0) = \ln 1 = 0$. Thus

$$L_{(1,0)}(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) = 0 + 2(x - 1) + 0(y - 0) = 2x - 2$$

i.e., $\ln(x^2 - y^2) = f(x, y) \approx L_{(1,0)}(x, y) = 2x - 2$ for (x, y) near $(1, 0)$.

By "near" $(1, 0)$ we mean a disk centred at $(1, 0)$. How do we find such disk? The domain of $f(x, y) = \ln(x^2 - y^2)$ is given by $x^2 - y^2 > 0$. Solving $x^2 - y^2 = 0$ we get $y = \pm x$. The two lines, $y = x$ and $y = -x$, divide the xy-plane into four regions. By using test points we figure out that the shaded region in the figure below is the domain of f. We can use any disk centred at $(1, 0)$ which is contained within this domain.



9. Let $f(x,y) = \sqrt{x^2+4y^2} = (x^2+4y^2)^{1/2}$. Note that f is defined and continuous near $(3,2)$. The partial derivatives of f are:

$$f_x(x,y) = \frac{1}{2}(x^2+4y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+4y^2}} \quad \text{and} \quad f_x(3,2) = \frac{3}{\sqrt{9+16}} = \frac{3}{5}$$
$$f_y(x,y) = \frac{1}{2}(x^2+4y^2)^{-1/2}(8y) = \frac{4y}{\sqrt{x^2+4y^2}} \quad \text{and} \quad f_y(3,2) = \frac{8}{5}$$

Since f_x and f_y are continuous near $(3,2)$, we conclude that f is differentiable at $(3,2)$, and therefore $f(x,y) \approx L_{(3,2)}(x,y)$ for (x,y) near $(3,2)$. We compute $f(3,2) = \sqrt{9+16} = 5$. Thus,

$$\begin{aligned} L_{(3,2)}(x,y) &= f(3,2) + f_x(3,2)(x-3) + f_y(3,2)(y-2) \\ &= 5 + \frac{3}{5}(x-3) + \frac{8}{5}(y-2) \\ &= \frac{3}{5}x + \frac{8}{5}y \end{aligned}$$

We conclude that

$$\sqrt{x^2+4y^2} = f(x,y) \approx L_{(3,2)}(x,y) = \frac{3}{5}x + \frac{8}{5}y$$

for (x,y) near $(3,2)$.

By "near" $(3,2)$ we mean a disk centred at $(3,2)$. Note that f is defined at all $(x,y) \in \mathbb{R}^2$, and the partial derivatives f_x and f_y are continuous as long as $(x,y) \neq (0,0)$. Thus, we can use any disk centred at $(3,2)$ which is small enough, i.e., which does not contain $(0,0)$.

11. The function $f(x,y) = x^2ye^y - 2$ is defined on all of $\mathbb{R}^2$. The partial derivatives $f_x(x,y) = 2xye^y$ and $f_y(x,y) = x^2e^y + x^2ye^y$ are continuous at all (x,y), and therefore f is differentiable at all $(x,y) \in \mathbb{R}^2$.

To be more precise: pick a point (a,b) in $\mathbb{R}^2$ and (in this case) any open disk $B_r(a,b)$ centred at (a,b). The function f is defined on $B_r(a,b)$, and the partial derivatives f_x and f_y are continuous on $B_r(a,b)$. Using Theorem 6, we conclude that f is differentiable at (a,b).

13. The function $f(x,y) = x^2y - y^3$ is defined on all of $\mathbb{R}^2$, and thus on any disk around $(1,5)$ that we choose. The partial derivatives $f_x(x,y) = 2xy$ and $f_y(x,y) = x^2 - 3y^2$ are continuous on $\mathbb{R}^2$. Thus, using Theorem 6 with any disk centred at $(1,5)$, we conclude that f is differentiable.

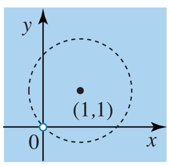
From $f(1,5) = 5 - 5^3 = -120$, $f_x(1,5) = 10$, and $f_y(1,5) = -74$ we obtain

$$\begin{aligned} L_{(1,5)}(x,y) &= f(1,5) + f_x(1,5)(x-1) + f_y(1,5)(y-5) \\ &= -120 + 10(x-1) - 74(y-5) \\ &= 240 + 10x - 74y \end{aligned}$$

15. Since $x^2+y^2 \geq 0$, and $x^2+y^2 = 0$ if any only if $x = y = 0$, we conclude that the function $f(x,y) = \ln(x^2+y^2)$ is not defined at $(0,0)$ only. The partial derivatives

$$f_x(x,y) = \frac{1}{x^2+y^2}2x = \frac{2x}{x^2+y^2} \quad \text{and} \quad f_y(x,y) = \frac{1}{x^2+y^2}2y = \frac{2y}{x^2+y^2}$$

are defined and continuous at all (x,y), except at $(0,0)$. To prove differentiability of f at $(1,1)$, according to Theorem 6, we need an open disk centred at $(1,1)$ which does not contain $(0,0)$. The largest such disk has the radius of $\sqrt{2}$, equal to the distance from $(1,1)$ to $(0,0)$; see the figure below.



17. From $f(1,0) = -1$, $f_x = 4x - y$, $f_x(1,0) = 4$, and $f_y = -x + 12y^2$, $f_y(1,0) = -1$, we get the tangent plane equation

$$\begin{aligned} z &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) \\ z &= -1 + 4(x-1) + (-1)(y-0) \\ z &= 4x - y - 5 \end{aligned}$$

19. From $g(-2,2) = 6e^0 = 6$, $g_x = 3ye^{x+y}$, $g_x(-2,2) = 6$, and $g_y = 3e^{x+y} + 3ye^{x+y}$, $g_y(-2,2) = 9$, we get the tangent plane equation

$$\begin{aligned} z &= g(-2,2) + g_x(-2,2)(x-(-2)) + g_y(-2,2)(y-2) \\ z &= 6 + 6(x+2) + 9(y-2) \\ z &= 6x + 9y \end{aligned}$$

21. From $f(0,0) = \ln 4$,

$$\begin{aligned} f_x &= 1 + \frac{1}{x^2+y+4}\,2x = 1 + \frac{2x}{x^2+y+4} \quad \text{and} \quad f_x(0,0) = 1 \\ f_y &= \frac{1}{x^2+y+4} \quad \text{and} \quad f_y(0,0) = \frac{1}{4} \end{aligned}$$

we compute the equation of the tangent plane:

$$\begin{aligned} z &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) \\ &= \ln 4 + 1x + \frac{1}{4}y \\ &= x + \frac{1}{4}y + \ln 4 \end{aligned}$$

23. From $f(0,1) = 1$, $f_x = ye^{-x^2}(-2x)$, $f_x(0,1) = 0$, and $f_y = e^{-x^2}$, $f_y(0,1) = 1$, we compute the linear approximation

$$L_{(0,1)}(x,y) = f(0,1) + f_x(0,1)(x-0) + f_y(0,1)(y-1) = 1 + 0x + 1(y-1) = y$$

Thus, $f(-0.1, 0.9) \approx L_{(0,1)}(-0.1, 0.9) = 0.9$. (A calculator value is $f(-0.1, 0.9) = 0.9e^{-(0.1)^2} \approx 0.89104485$.)

25. Let $f(x,y) = \sqrt{x^2+y^2}$. Since 2.98 is close to 3 and 4.04 is close to 4, we will use the linear approximation $L_{(3,4)}(x,y)$. From $f(3,4) = \sqrt{3^2+4^2} = 5$,

$$\begin{aligned} f_x &= \frac{1}{2}(x^2+y^2)^{-1/2}\,2x = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad f_x(3,4) = \frac{3}{5} \\ f_y &= \frac{y}{\sqrt{x^2+y^2}} \quad \text{and} \quad f_y(3,4) = \frac{4}{5} \end{aligned}$$

we compute

$$\begin{aligned} L_{(3,4)}(x,y) &= f(3,4) + f_x(3,4)(x-3) + f_y(3,4)(y-4) \\ &= 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4) \\ &= \frac{3}{5}x + \frac{4}{5}y \end{aligned}$$

Therefore,

$$\sqrt{2.98^2 + 4.04^2} = f(2.98, 4.04) \approx L_{(3,4)}(2.98, 4.04) = \frac{3}{5}(2.98) + \frac{4}{5}(4.04) = 5.02$$

The calculator value is 5.020159360.

27. Let $f(x,y) = xe^y$. Since 2.98 is close to 3 and -0.04 is close to 0, we will use the linear approximation $L_{(3,0)}(x,y)$. From $f(3,0) = 3e^0 = 3$, $f_x = e^y$, $f_x(3,0) = 1$, and $f_y = xe^y$, $f_y(3,0) = 3$, we compute

$$\begin{aligned} L_{(3,0)}(x,y) &= f(3,0) + f_x(3,0)(x-3) + f_y(3,0)(y-0) \\ &= 3 + 1(x-3) + 3(y-0) \\ &= x + 3y \end{aligned}$$

Therefore,

$$2.98e^{-0.04} = f(2.98,-0.04) \approx L_{(3,0)}(2.98,-0.04) = 2.98 + 3(-0.04) = 2.87$$

The calculator value is 2.863152529.

29. We can answer this question with or without differentials; we will do both. Without differentials: using $S(m,h) = 0.20247m^{0.425}h^{0.725}$, we get

$$\begin{aligned} S(1.03m, 1.02h) &= 0.20247(1.03m)^{0.425}(1.02h)^{0.725} \\ &= 0.20247(1.03)^{0.425}m^{0.425}(1.02)^{0.725}h^{0.725} \\ &= (1.03)^{0.425}(1.02)^{0.725}\left(0.20247m^{0.425}h^{0.725}\right) \\ &\approx 1.02728S(m,h) \end{aligned}$$

Since $S(1.03m, 1.02h) = 1.02728S(m,h) = S(m,h) + 0.02728S(m,h)$, we conclude that the surface area increases by 2.728%.

Using differentials: from $S(m,h) = 0.20247m^{0.425}h^{0.725}$ we find

$$dS = 0.20247(0.425)m^{-0.575}h^{0.725}dm + 0.20247m^{0.425}(0.725)h^{-0.275}dh$$

Let $\Delta S = S(1.03m, 1.02h) - S(m,h)$. Then $\Delta S \approx dS$, where the differential dS is calculated with $dm = 0.03m$ and $dh = 0.02h$:

$$\begin{aligned} dS &= 0.20247(0.425)m^{-0.575}h^{0.725}(0.03m) + 0.20247m^{0.425}(0.725)h^{-0.275}(0.02h) \\ &= (0.425)(0.03)0.20247m^{0.425}h^{0.725} + (0.725)(0.02)0.20247m^{0.425}h^{0.725} \\ &= (0.02725)(0.20247m^{0.425}h^{0.725}) \\ &= 0.02725S(m,h) \end{aligned}$$

Thus, S increases by approximately 2.725%.

When $m = 75$ kg and $h = 1.72$ m, then $S(75, 1.72) = 1.87940$ m^2 and $S(1.03 \cdot 75, 1.02 \cdot 1.72) = 1.93068$ m^2. The differential $dS = 0.02725S(75, 1.72) = 0.05121$ is a good approximation of the true increase $S(1.03 \cdot 75, 1.02 \cdot 1.72) - S(75, 1.72) = 1.93068 - 1.87940 = 0.05128$.

31. The increase in the volume ΔV is approximated by the differential dV, given by [for the purpose of differentiation, write $V = ab(a+b)\pi/12 = \frac{\pi}{12}(a^2b + ab^2)$]

$$dV = V_a da + V_b db = \frac{\pi}{12}(2ab + b^2)da + \frac{\pi}{12}(a^2 + 2ab)db$$

In particular, when $a = 6$ mm, $b = 4.4$ mm, $da = 0.2$ mm, and $db = 0.3$ mm, we get

$$\begin{aligned} dV &= \frac{\pi}{12}\left[2(6)(4.4) + (4.4)^2\right](0.2) + \frac{\pi}{12}\left[(6)^2 + 2(6)(4.4)\right](0.3) \\ &= \frac{\pi}{12}14.432 + \frac{\pi}{12}26.64 \\ &\approx 10.75262 \end{aligned}$$

Thus, the increase in volume is approximately 10.8 mm^3.

33. The volume of a cylinder of radius r and height h is given by $V(r,h) = \pi r^2 h$. The differential of V is $dV = 2\pi rh dr + \pi r^2 dh$. Given that $dr = 0.025r$ and $dh = 0.025h$, we get

$$dV = 2\pi rh(0.025r) + \pi r^2(0.025h) = 0.05\pi r^2 h + 0.025\pi r^2 h = 0.075\pi r^2 h = 0.075V(r,h)$$

Thus, the volume is calculated with an approximate error of 7.5%.

35. The change in the function Δf can be approximated by

$$\Delta f \approx df = 2cxy^3 dx + 3cx^2y^2 dy$$

It is given that $dx = -0.02x$ and $dy = 0.02y$. Thus,

$$df = 2cxy^3(-0.02x) + 3cx^2y^2(0.02y) = cx^2y^3(-0.04 + 0.06) = 0.02cx^2y^3 = 0.02f(x, y)$$

Thus, f increases by approximately 2%.

37. To compute the partial derivatives we need to use the definition:

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{0 - 0}{h} = \lim_{h\to 0} 0 = 0$$

Likewise,

$$f_y(0, 0) = \lim_{h\to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h\to 0} \frac{0 - 0}{h} = \lim_{h\to 0} 0 = 0$$

It is given that $f(0, 0) = 0$. Thus, the linearization of f is

$$L_{(0,0)}(x, y) = f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0) = 0 + 0x + 0y = 0$$

Note that if $y = x$, then $f(x, y) = f(x, x) = x^2/2x^2 = 1/2$; i.e., along the line $y = x$ (as long as $x \neq 0$) the value of f is $1/2$. So no matter how close to $(0, 0)$ we get, i.e., no matter how small an open disk around $(0, 0)$ we take, the function assumes the value of $1/2$ at some point in it; thus, $L_{(0,0)}(x, y) = 0$ is not a good approximation of f.

We conclude that the partial derivatives of f are not continuous at $(0, 0)$. If they were, f would be differentiable, and $L_{(0,0)}(x, y) = 0$ would have been a good approximation of f near $(0, 0)$.

Section 6 The Chain Rule

1. Using the chain rule,

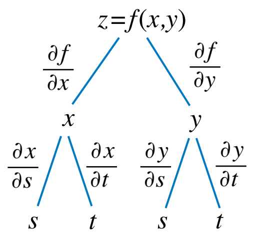
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

An alternative way to write this formula is to replace the variables by their function names:

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial h}{\partial s}$$

Likewise,

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial h}{\partial t}$$



3. Using the chain rule for $E = f(M(v), N(v))$, we get

$$\frac{dE}{dv} = \frac{\partial E}{\partial M}\frac{dM}{dv} + \frac{\partial E}{\partial N}\frac{dN}{dv}$$

5. From $y = 1/x$ we get $xy - 1 = 0$. Let $F(x, y) = xy - 1$. Since F is a polynomial, its partial derivatives are polynomials, and hence continuous. Thus, F is differentiable function, and the set $F(x, y) = xy - 1 = 0$ represents the hyperbola $y = 1/x$.

7. From $F(x, y) = x - 2y^2 - 1 = 0$ we obtain $x = 2y^2 - 1$. Might be easier if we switch axes: the curve $y = 2x^2 - 1$ is a parabola which opens toward the positive y-axis, with the vertex at $(0, -1)$, and the x-intercepts at $x = \pm 1/\sqrt{2}$. Thus, $x = 2y^2 - 1$ is a parabola which opens towards the positive x-axis; its vertex is at $(-1, 0)$ on the y-axis, and the y-intercepts are $y = \pm 1/\sqrt{2}$.

9. From $F(x, y) = x^2 y - 1 = 0$ we obtain $y = 1/x^2$; this is a hyperbola in the first and the second quadrants, whose asymptotes are the x-axis and the positive y-axis.

11. From $F(x, y) = x^2 - y^2 = 0$ we get $y^2 = x^2$ and $y = \pm x$. Thus, $F(x, y) = 0$ represents the pair of intersecting lines $y = x$ and $y = -x$.

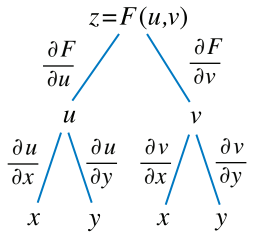
13. To understand how to apply the chain rule, we think of $z = F(g(x, y), h(x, y))$ as $z = F(u, v)$, where $u = g(x, y)$ and $v = h(x, y)$. Using the tree diagram below, we write the chain rule in two ways, by using the variables u and v, and then by replacing them with the functions they represent:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial g}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial h}{\partial x}$$

Likewise,

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial F}{\partial u}\frac{\partial g}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial h}{\partial y}$$

Note that z and F represent the same function, when viewed as functions of u and v; so $\partial z/\partial u = \partial F/\partial u$ and $\partial z/\partial v = \partial F/\partial v$.



15. Using the chain rule,
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (3x^2 - 2y)(\cos t) + (-2x)(5) = (3\sin^2 t - 10t)\cos t - 10\sin t$$

17. Write $z = \sqrt{p^2 - 5q - 2} = (p^2 - 5q - 2)^{1/2}$. By the chain rule,
$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial p}\frac{dp}{dt} + \frac{\partial z}{\partial q}\frac{dq}{dt}\\ &= \frac{1}{2}(p^2 - 5q - 2)^{-1/2}(2p)(3t^2) + \frac{1}{2}(p^2 - 5q - 2)^{-1/2}(-5)(1)\\ &= \frac{1}{2\sqrt{p^2 - 5q - 2}}\left(6pt^2 - 5\right) = \frac{6t^5 - 5}{2\sqrt{t^6 - 5t - 2}}\end{aligned}$$

19. By the chain rule,
$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\\ &= -y^2e^{-x}(2t) + 2ye^{-x}\left(-\frac{1}{t^2}\right)\\ &= -2ye^{-x}\left(yt + \frac{1}{t^2}\right) = -\frac{2}{t}e^{-t^2}\left(1 + \frac{1}{t^2}\right)\end{aligned}$$

21. We think of the number of whales as the function $N = N(P, T)$, where $P = P(t)$ and $T = T(t)$. Using the chain rule,
$$\frac{dN}{dt} = \frac{\partial N}{\partial P}\frac{dP}{dt} + \frac{\partial N}{\partial T}\frac{dT}{dt}$$
The derivative dN/dt is the rate of change of the number of whales with respect to time. The partial derivative $\partial N/\partial P$ describes the rate of change of the number of whales due to the availability of plankton (assuming that there is no change in ocean temperature), and $\partial N/\partial T$ gives the rate of change of the number of whales with respect to the ocean temperature (under the assumption that the availability of plankton remains unchanged). The derivative dP/dt describes how the availability of plankton changes over time, and dT/dt is the rate of change in ocean temperature with respect to time.

23. Let's consider the term $h = g(x + y^2)$ by itself. It helps if we think of it as the composition $h = g \circ v$, where $v = x + y^2$. Thus,
$$\frac{\partial g(x + y^2)}{\partial x} = \frac{\partial h}{\partial x} = \frac{dh}{dv}\frac{\partial v}{\partial x} = h'(v) \cdot 1 = g'(x + y^2)$$
As practice (we will not need it here):
$$\frac{\partial g(x + y^2)}{\partial y} = \frac{\partial h}{\partial y} = \frac{dh}{dv}\frac{\partial v}{\partial y} = h'(v) \cdot (2y) = 2yg'(x + y^2)$$

Differentiating $f(x,y) = g(x) + g(x+y^2)$. with respect to x, we get

$$\frac{\partial f}{\partial x} = g'(x) + g'(x+y^2)$$

Thus, $(\partial f/\partial x)(0,0) = g'(0) + g'(0) = 10$. (The condition $g(0) = 4$ (which appears in the first printing of this book) is not needed.)

25. Using the chain rule,

$$\begin{aligned} w'(t) &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \\ &= 6xe^{y^2}(-\sin t) + (3x^2-4)e^{y^2}(2y)(\cos t) \\ &= 6\cos t e^{\sin^2 t}(-\sin t) + (3\cos^2 t - 4)e^{\sin^2 t}(2\sin t)\cos t \end{aligned}$$

No need to simplify any further; since $\cos t$ appears in both factors, it follows that $w'(\pi/2) = 0$.

27. Using the product and the chain rules,

$$u_x(x,t) = 2xt^3 f(x-5t) + x^2t^3 f'(x-5t)(1) = 2xt^3 f(x-5t) + x^2t^3 f'(x-5t)$$

and

$$u_t(x,t) = x^2(3t^2) f(x-5t) + x^2t^3 f'(x-5t)(-5) = 3x^2t^2 f(x-5t) - 5x^2t^3 f'(x-5t)$$

29. Using the chain rule,

$$\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial a}\frac{\partial a}{\partial u} + \frac{\partial z}{\partial b}\frac{\partial b}{\partial u} \\ &= \frac{1}{a+b^3-2}(1)(v) + \frac{1}{a+b^3-2}(3b^2)(2u) \\ &= \frac{1}{a+b^3-2}\left(v + 6b^2u\right) = \frac{v + 6(u^2-v^2)^2u}{uv + (u^2-v^2)^3 - 2} \end{aligned}$$

Likewise,

$$\begin{aligned} \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial a}\frac{\partial a}{\partial v} + \frac{\partial z}{\partial b}\frac{\partial b}{\partial v} \\ &= \frac{1}{a+b^3-2}(1)(u) + \frac{1}{a+b^3-2}(3b^2)(-2v) \\ &= \frac{1}{a+b^3-2}\left(u - 6b^2v\right) = \frac{u - 6(u^2-v^2)^2v}{uv + (u^2-v^2)^3 - 2} \end{aligned}$$

31. By the chain rule,

$$\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\ &= y\cos(xy)(2uv) + x\cos(xy)(-2v^4) \\ &= \cos(xy)\left(2yuv - 2xv^4\right) \\ &= \cos(-2u^3v^5)\left(-4u^2v^5 - 2u^2v^5\right) = -6u^2v^5\cos(-2u^3v^5) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\ &= y\cos(xy)(u^2) + x\cos(xy)(-8uv^3) \\ &= \cos(xy)\left(yu^2 - 8xuv^3\right) \\ &= \cos(-2u^3v^5)\left(-2u^3v^4 - 8u^3v^4\right) = -10u^3v^4\cos(-2u^3v^5) \end{aligned}$$

33. Using the quotient and the chain rules,

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\\ &= \frac{2x(1-xy)-(x^2-y)(-y)}{(1-xy)^2}(2v) + \frac{(-1)(1-xy)-(x^2-y)(-x)}{(1-xy)^2}(0)\\ &= \frac{(2x-x^2y-y^2)2v}{(1-xy)^2} = \frac{2v\left(4uv-20u^2v^3-25v^2\right)}{(1-10uv^2)^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\\ &= \frac{2x(1-xy)-(x^2-y)(-y)}{(1-xy)^2}(2u) + \frac{(-1)(1-xy)-(x^2-y)(-x)}{(1-xy)^2}(5)\\ &= \frac{2u(2x-x^2y-y^2)}{(1-xy)^2} + \frac{5(-1+x^3)}{(1-xy)^2}\\ &= \frac{2u\left(4uv-20u^2v^3-25v^2\right)+5(-1+8u^3v^3)}{(1-10uv^2)^2}\end{aligned}$$

35. From $3x^2 + 2yy' = 0$ we get $y' = -3x^2/2y$.

37. Using the chain and the product rules in differentiating the first term, we get

$$\begin{aligned}e^{xy}(y+xy') - 2yy' &= 0\\ (xe^{xy}-2y)y' &= -ye^{xy}\\ y' &= -\frac{ye^{xy}}{xe^{xy}-2y}\end{aligned}$$

39. We compute

$$\begin{aligned}\cos(2x-y)(2-y') &= -\sin(x-3y)(1-3y')\\ (-\cos(2x-y)-3\sin(x-3y))y' &= -2\cos(2x-y)-\sin(x-3y)\\ y' &= \frac{2\cos(2x-y)+\sin(x-3y)}{\cos(2x-y)+3\sin(x-3y)}\end{aligned}$$

41. Keeping y constant and differentiating with respect to x using the product and the chain rules, we get

$$(1)yz + xy(1)\frac{\partial z}{\partial x} = 0 \quad \text{and} \quad yz + xy\frac{\partial z}{\partial x} = 0$$

Thus, $\partial z/\partial x = -yz/xy = -z/x$. Likewise, keeping x constant and differentiating with respect to y,

$$x(1)z + xy(1)\frac{\partial z}{\partial y} = 0 \quad \text{and} \quad xz + xy\frac{\partial z}{\partial x} = 0$$

Thus, $\partial z/\partial x = -z/y$.

43. Differentiating $e^{xz} - eyz = 0$ with respect to x while keeping y constant, we obtain

$$\begin{aligned}e^{xz}\left(z + x\frac{\partial z}{\partial x}\right) - ey\frac{\partial z}{\partial x} &= 0\\ (xe^{xz}-ey)\frac{\partial z}{\partial x} &= -ze^{xz}\\ \frac{\partial z}{\partial x} &= -\frac{ze^{xz}}{xe^{xz}-ey}\end{aligned}$$

Differentiating $e^{xz} - eyz = 0$ with respect to y, while keeping x fixed, we obtain

$$e^{xz} x \frac{\partial z}{\partial y} - ez - ey \frac{\partial z}{\partial y} = 0$$
$$(xe^{xz} - ey) \frac{\partial z}{\partial y} = ez$$
$$\frac{\partial z}{\partial y} = \frac{ez}{xe^{xz} - ey}$$

45. From $(x^2+y^2)^2 - x^2 + y^2 = 0$ we get $x^4 + 2x^2y^2 + y^4 - x^2 + y^2 = 0$, and $y^4 + (2x^2+1)y^2 + x^4 - x^2 = 0$. Using the quadratic formula gives

$$y^2 = \frac{-2x^2 - 1 \pm \sqrt{4x^4 + 4x^2 + 1 - 4x^4 + 4x^2}}{2} = \frac{-2x^2 - 1 \pm \sqrt{8x^2 + 1}}{2}$$

To make sure that the expression for y^2 is positive, we must take the plus sign, i.e.,

$$y^2 = \frac{-2x^2 - 1 + \sqrt{8x^2 + 1}}{2}$$

Consequently,

$$y = \pm \frac{1}{\sqrt{2}} \sqrt{-2x^2 - 1 + \sqrt{8x^2 + 1}} = \pm \frac{1}{2} \sqrt{-4x^2 - 2 + 2\sqrt{8x^2 + 1}}$$

In the last step, we multiplied and divided by $\sqrt{2}$, and brought the $\sqrt{2}$ from the numerator inside the square root.

47. Suppose that the two parallel lines are $x = 1$ and $x = 2$. Then $x - 1 = 0$ and $x - 2 = 0$; so, let $F(x, y) = (x - 1)(x - 2) = x^2 - 3x + 2$. The set $F(x, y) = 0$ consists of the lines $x = 1$ and $x = 2$.

In the same way we can construct infinitely many functions with the required property. For instance, the set defined by $F(x, y) = 0$, where $F(x, y) = (x + 2y - 4)(x + 2y + 3)$, consists of parallel lines $x + 2y - 4 = 0$ and $x + 2y + 3 = 0$. (Note that F is a polynomial, and therefore has continuous partial derivatives.)

Section 7 Second-Order Partial Derivatives and Applications

1. Assume that $f = f(x, y)$; the partial derivative f_y is also a function of x and y, and so

$$f_{yy}(2,7) = (f_y)_y(2,7) = \lim_{h\to 0} \frac{f_y(2,7+h) - f_y(2,7)}{h}$$

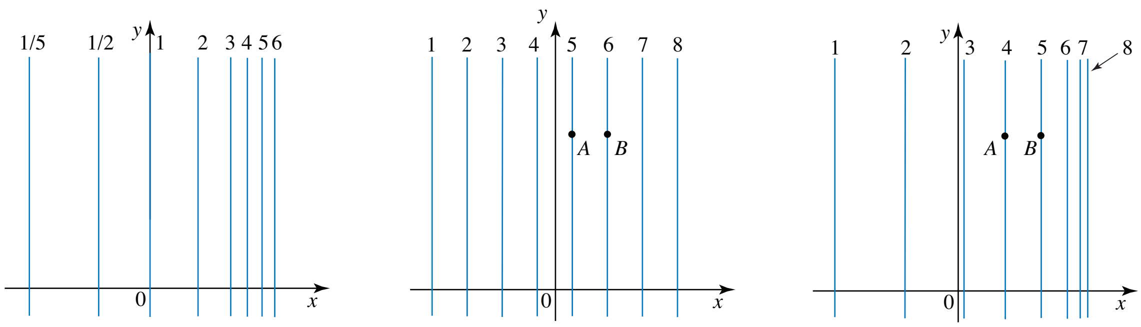
If needed, we can further write the first derivatives as limits:

$$f_y(2,7+h) = \lim_{h_1\to 0} \frac{f(2,7+h+h_1) - f(2,7+h)}{h_1} \quad \text{and} \quad f_y(2,7) = \lim_{h_2\to 0} \frac{f(2,7+h_2) - f(2,7)}{h_2}$$

3. Start with a simple case; if $f(x, y) = e^x$, then $f_{xx}(x, y) = e^x > 0$ for all $(x, y) \in \mathbb{R}^2$. What do level curves of f look like? From $f(x, y) = e^x = c$ we get $x = \ln c$; the level curves are vertical lines (see below, left). Note that they get closer to each other as c advances by 1. (The x-intercepts of the level curves are (from left to right): $\ln(1/5) \approx -1.61$, $\ln(1/2) \approx -0.69$, $\ln 1 = 0$, $\ln 2 = 0.69$, $\ln 3 = 1.10$, $\ln 4 = 1.39$, $\ln 5 = 1.61$.)

Let's think a bit more about this: $f_{xx} > 0$ means that $(f_x)_x > 0$, i.e., the partial derivative of f_x in the direction of the positive x-axis is increasing. Consider the contour diagram below, centre (as suggested by the diagram, level curves are evenly spaced). The rate of change of f in the x-direction is positive at all points; in particular $f_x(A) > 0$ and $f_x(B) > 0$, where B is a point near A, with the same y-coordinate at A. However, at all points the rates of increase in f in the x direction are the same; thus $f_x(A) = f_x(B)$, and consequently, $f_{xx}(A) = 0$.

To make f_x increase in the horizontal direction, we make the distance between the level curves smaller as we move along the x-axis (see the figure below, right; thus, we have arrived at the idea suggested by the contour diagram of $f(x, y) = e^x$). To check (figure below, right): the values of f increase as we move toward the right, thus $f_x(A) > 0$ and $f_x(B) > 0$. This time, however, $f_x(A) < f_x(B)$, since the same change in the function (of 1 unit) occurs over the smaller distance at B. Thus, f_x is increasing the the x-direction, i.e., $f_{xx} > 0$.



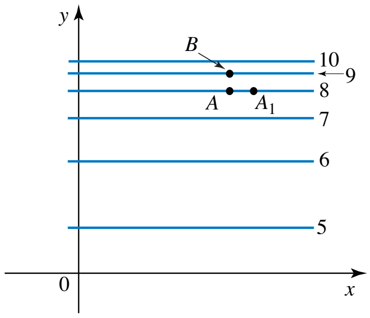
5. No, because the term $-xy^2$ is of degree 3.

7. For instance, $f(x, y) = 2 + (x - 2)^2 + (x - 2)(y - 1) + (x - 2)^4$. Note that the extra term $h(x, y) = (x - 2)^4$ contributes nothing to the degree-2 Taylor polynomial: its value, and the values of its partial derivatives are either identically zero, or evaluate to zero at the point $(2, 1)$. More precisely: $h = (x - 2)^4$, $h_x = 4(x - 2)^3$, $h_{xx} = 12(x - 2)^2$ all evaluate to zero at $(2, 1)$; the remaining derivatives are $h_y = 0$, $h_{xy} = 0$ and $h_{yy} = 0$.

Expanding this idea, we realize that any function of the form $f(x, y) = 2 + (x - 2)^2 + (x - 2)(y - 1) + g(x, y)$, where $g(x, y)$ is a polynomial in $x - 2$ and $y - 1$ all of whose terms are of order three or higher, has the degree-2 Taylor polynomial equal to $2 + (x - 2)^2 + (x - 2)(y - 1)$. (As above, we can argue that $g(x, y)$, and all its first and second order partial derivatives vanish at $(2, 1)$.)

9. The level curves of f are horizontal. In other words, f does not change in the direction of the x-axis, and so $f_x = 0$ at all points in $\mathbb{R}^2$; in particular, $f_x(A) = 0$ and $f_x(A_1) = 0$, where A_1 is near A and lies on the same level curve as A. From this, we conclude that $f_{xx}(A) = 0$.

In the vertical direction away from A, the values of f increase; thus, $f_y(A) > 0$. Pick a point B near A and vertically above it (as shown in figure below). Since $f_x(A) = 0$ and the value of f_x at a nearby point B is also zero, it follows that $f_{xy}(A) = 0$. What is $f_{yy}(A) = (f_y)_y(A)$? f increases in the vertical direction, thus $f_y(A) > 0$ and $f_y(B) > 0$. However, $f_y(B) > f_y(A)$, since the same change in f happens over a smaller distance at B. Thus, f_y is increasing in the vertical direction, and $f_{yy}(A) > 0$.



11. Thinking of $f_{xy}(4,1)$ as $(f_x)_y(4,1)$, we write

$$(f_x)_y(4,1) = \lim_{h\to 0} \frac{f_x(4,1+h) - f_x(4,1)}{h} \approx \frac{f_x(4,1+h) - f_x(4,1)}{h}$$

The smallest values of h we can use are $h = 1$ and $h = -1$. Thus, to estimate $(f_x)_y(4,1)$, we need to know $f_x(4,1)$, $f_x(4,2)$ and $f_x(4,0)$. In this exercise, we will use two estimates for each derivative, and then take the average. To start:

$$f_x(4,1) \approx \frac{f(4+h,1) - f(4,1)}{h}$$

When $h = 1$,

$$f_x(4,1) \approx \frac{f(5,1) - f(4,1)}{1} = \frac{3.1 - 3.5}{1} = -0.4$$

When $h = -1$,

$$f_x(4,1) \approx \frac{f(3,1) - f(4,1)}{-1} = \frac{3.2 - 3.5}{-1} = 0.3$$

Thus, $f_x(4,1) \approx (-0.4 + 0.3)/2 = -0.05$. We repeat the same routine for the remaining partial derivatives:

$$f_x(4,0) \approx \frac{f(4+h,0) - f(4,0)}{h}$$

When $h = 1$,

$$f_x(4,0) \approx \frac{f(5,0) - f(4,0)}{1} = \frac{2.9 - 2.8}{1} = 0.1$$

When $h = -1$,

$$f_x(4,0) \approx \frac{f(3,0) - f(4,0)}{-1} = \frac{2.3 - 2.8}{-1} = 0.5$$

Taking the average, $f_x(4,0) \approx (0.1 + 0.5)/2 = 0.3$. As well,

$$f_x(4,2) \approx \frac{f(4+h,2) - f(4,2)}{h}$$

When $h = 1$,

$$f_x(4,2) \approx \frac{f(5,2) - f(4,2)}{1} = \frac{3.4 - 3.3}{1} = 0.1$$

When $h = -1$,

$$f_x(4,2) \approx \frac{f(3,2) - f(4,2)}{-1} = \frac{3.2 - 3.3}{-1} = 0.1$$

Taking the average, $f_x(4,2) \approx 0.1$. We are now ready to estimate $(f_x)_y(4,1)$. When $h = 1$,

$$(f_x)_y(4,1) \approx \frac{f_x(4,2) - f_x(4,1)}{1} \approx \frac{0.1 - (-0.05)}{1} = 0.15$$

When $h = -1$,

$$(f_x)_y(4,1) \approx \frac{f_x(4,0) - f_x(4,1)}{-1} \approx \frac{0.3 - (-0.05)}{-1} = 0.35$$

Taking the average, $(f_x)_y(4,1) \approx (0.15 + 0.35)/2 = 0.25$.

Obviously, this is a fairly long procedure. To make it shorter, instead of calculating both forward ($h = 1$) and backward ($h = -1$) difference quotients, we usually use forward quotients only.

13. We will use forward difference quotients only (see the comment made at the end of the solution of Exercise 11). To approximate $f_y(5,1)$ we use

$$f_y(5,1) = \lim_{h\to 0} \frac{f(5,1+h) - f(5,1)}{h} \approx \frac{f(5,1+h) - f(5,1)}{h}$$

(Forward difference quotient means that we take $h > 0$.) Taking $h = 1$,

$$f_y(5,1) \approx \frac{f(5,2) - f(5,1)}{1} = \frac{3.4 - 3.1}{1} = 0.3$$

As well,

$$f_y(6,1) \approx \frac{f(6,2) - f(6,1)}{1} = \frac{3.3 - 2.8}{1} = 0.5$$

Now

$$(f_y)_x(5,1) \approx \frac{f_y(5+h,1) - f_y(5,1)}{h}$$

When $h = 1$,

$$(f_y)_x(5,1) \approx \frac{f_y(6,1) - f_y(5,1)}{h} = \frac{0.5 - 0.3}{1} = 0.2$$

15. Using the chain rule, we compute

$$\begin{aligned}
u_x &= f'(x - st)(1) = f'(x - st)\\
u_t &= f'(x - st)(-s) = -sf'(x - st)\\
u_{xx} &= f''(x - st)(1) = f''(x - st)\\
u_{tx} &= -sf''(x - st)(1) = -sf''(x - st)\\
u_{tt} &= -sf''(x - st)(-s) = s^2 f''(x - st)
\end{aligned}$$

17. Write $z = \sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2}$. Note that z is symmetric in x and y (i.e., interchanging x and y does not change z). This means that once we compute f_x, we obtain f_y by interchanging x and y. Ditto for f_{xx} and f_{yy}. By the chain rule,

$$\begin{aligned}
z_x &= \frac{1}{2}(x^2 + y^2)^{-1/2}(2x) = x(x^2 + y^2)^{-1/2}\\
z_y &= y(x^2 + y^2)^{-1/2}\\
z_{xx} &= (x^2 + y^2)^{-1/2} + x\left(-\frac{1}{2}\right)(x^2 + y^2)^{-3/2}(2x)\\
&= (x^2 + y^2)^{-3/2}[x^2 + y^2 - x^2] = \frac{y^2}{(x^2 + y^2)^{3/2}}\\
z_{yy} &= \frac{x^2}{(x^2 + y^2)^{3/2}}
\end{aligned}$$

$$z_{xy} = x\left(-\frac{1}{2}\right)(x^2+y^2)^{-3/2}(2y) = -\frac{xy}{(x^2+y^2)^{3/2}}$$

19. This is straightforward: $f_x = -1$, $f_y = 0$, $f_{xx} = 0$, $f_{xy} = f_{yx} = 0$ and $f_{yy} = 0$.

21. We compute $g_x = y(-1)x^{-2} = -yx^{-2}$, $g_y = x^{-1}$, $g_{xx} = -y(-2)x^{-3} = 2yx^{-3}$, $g_{xy} = -x^{-2}$, $g_{yx} = -x^{-2}$, and $g_{yy} = 0$.

23. Using the chain rule,

$$\begin{aligned}
z_x &= (-1)(1-xy)^{-2}(-y) = y(1-xy)^{-2} \\
z_{xx} &= y(-2)(1-xy)^{-3}(-y) = 2y^2(1-xy)^{-3} \\
z_{xy} &= (1-xy)^{-2} + y(-2)(1-xy)^{-3}(-x) \\
&= (1-xy)^{-3}[1-xy+2xy] = (1-xy)^{-3}(1+xy)
\end{aligned}$$

25. We use the fact that a degree-2 Taylor polynomial of a polynomial all of whose terms are of degree 3 or higher is zero. For instance, the degree-2 Taylor polynomials of $x^3+3x^2y^2$ and $2x^5-y^4+x^3y^2$ at $(0,0)$ are zero. (This can easily be checked by calculating the values of these polynomials and their first and second partial derivatives at $(0,0)$.) So, take any function $k(x,y)$. The degree-2 Taylor polynomials of both $f(x,y) = k(x,y)+x^3+3x^2y^2$ and $g(x,y) = k(x,y)+2x^5-y^4+x^3y^2$ are equal to the degree-2 Taylor polynomial of $k(x,y)$.

27. Note that f is a degree-2 polynomial; thus, it is equal to its degree-2 Taylor polynomial *at* $(0,0)$ (this fact is not true if the Taylor polynomial is based at any point other than the origin).

Alternatively, we compute the degree-2 Taylor polynomial from scratch: $f(x,y) = 1-x-x^2+3y^2$, $f(0,0) = 1$; $f_x = -1-2x$, $f_x(0,0) = -1$; $f_y = 6y$, $f_y(0,0) = 0$; $f_{xx} = -2$, $f_{xx}(0,0) = -2$; $f_{xy} = 0$, $f_{xy}(0,0) = 0$; and $f_{yy} = 6$, $f_{yy}(0,0) = 6$. Thus,

$$\begin{aligned}
T_2(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) \\
&\quad + \frac{1}{2}\left(f_{xx}(0,0)(x-0)^2 + 2f_{xy}(0,0)(x-0)(y-0) + f_{yy}(0,0)(y-0)^2\right) \\
&= 1 + (-1)x + (0)y + \frac{1}{2}\left((-2)x^2 + 2(0)xy + (6)y^2\right) \\
&= 1 - x - x^2 + 3y^2
\end{aligned}$$

29. Recall that $\sin t = t - t^3/3! + t^5/5! - \cdots$; thus, the degree-2 Taylor polynomial of $\sin t$ at $t = 0$ is $T_2(t) = t$. We conclude that the degree-2 Taylor polynomial of $f(x,y) = \sin(2x-y)$ at $(0,0)$ is $T_2(x,y) = \sin(2x-y)$.

Alternatively, we compute the degree-2 Taylor polynomial from scratch: $f(x,y) = \sin(2x-y)$, $f(0,0) = 0$; $f_x = 2\cos(2x-y)$, $f_x(0,0) = 2$; $f_y = -\cos(2x-y)$, $f_y(0,0) = -1$. All second partial derivatives involve $\sin(2x-y)$, and therefore all corresponding coefficients are zero. It follows that

$$\begin{aligned}
T_2(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) \\
&\quad + \frac{1}{2}\left(f_{xx}(0,0)(x-0)^2 + 2f_{xy}(0,0)(x-0)(y-0) + f_{yy}(0,0)(y-0)^2\right) \\
&= 0 + (2)x + (-1)y + \frac{1}{2}\left((0)x^2 + 2(0)xy + (0)y^2\right) \\
&= 2x - y
\end{aligned}$$

31. To keep track of the derivatives and their values, we use the table:

f and its partial derivatives	Value at $(1,0)$
$f = e^{-x}\sin y$	0
$f_x = -e^{-x}\sin y$	0
$f_y = e^{-x}\cos y$	e^{-1}
$f_{xx} = e^{-x}\sin y$	0
$f_{xy} = f_{yx} = -e^{-x}\cos y$	$-e^{-1}$
$f_{yy} = -e^{-x}\sin y$	0

Thus

$$\begin{aligned}T_2(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0)\\ &\qquad + \frac{1}{2}\left(f_{xx}(1,0)(x-1)^2 + 2f_{xy}(1,0)(x-1)(y-0) + f_{yy}(1,0)(y-0)^2\right)\\ &= 0 + 0(x-1) + e^{-1}y + \frac{1}{2}\left(0(x-1)^2 + 2(-e^{-1})(x-1)y + (0)y^2\right)\\ &= \frac{1}{e}y - \frac{1}{e}(x-1)y = \frac{1}{e}[y-(x-1)y]\end{aligned}$$

33. We use the table:

f and its partial derivatives	Value at $(0,0)$
$f = \ln(1-x^2-y^2)$	0
$f_x = \dfrac{-2x}{1-x^2-y^2}$	0
$f_y = \dfrac{-2y}{1-x^2-y^2}$	0
$f_{xx} = \dfrac{-2(1-x^2-y^2)+2x(-2x)}{1-x^2-y^2}$	-2
$f_{xy} = f_{yx} = \dfrac{-4xy}{1-x^2-y^2}$	0
$f_{yy} = \dfrac{-2(1-x^2-y^2)+2y(-2y)}{1-x^2-y^2}$	-2

Thus

$$\begin{aligned}T_2(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)\\ &\qquad + \frac{1}{2}\left(f_{xx}(0,0)(x-0)^2 + 2f_{xy}(0,0)(x-0)(y-0) + f_{yy}(0,0)(y-0)^2\right)\\ &= 0 + 0x + 0y + \frac{1}{2}\left((-2)x^2 + 2(0)xy + (-2)y^2\right)\\ &= -x^2 - y^2\end{aligned}$$

35. We compute:

$$\begin{aligned}f &= \sqrt{3x+y-1} = (3x+y-1)^{1/2}, \quad f(1,-1) = 1\\ f_x &= \frac{1}{2}(3x+y-1)^{-1/2}(3) = \frac{3}{2}(3x+y-1)^{-1/2}, \quad f_x(1,-1) = \frac{3}{2}\\ f_y &= \frac{1}{2}(3x+y-1)^{-1/2}(1) = \frac{1}{2}(3x+y-1)^{-1/2}, \quad f_y(1,-1) = \frac{1}{2}\end{aligned}$$

$$f_{xx} = \frac{3}{2}\left(-\frac{1}{2}\right)(3x+y-1)^{-3/2}(3) = -\frac{9}{4}(3x+y-1)^{-3/2}, \quad f_x(1,-1) = -\frac{9}{4}$$
$$f_{xy} = \frac{3}{2}\left(-\frac{1}{2}\right)(3x+y-1)^{-3/2}(1) = -\frac{3}{4}(3x+y-1)^{-3/2}, \quad f_x(1,-1) = -\frac{3}{4}$$
$$f_{yy} = \frac{1}{2}\left(-\frac{1}{2}\right)(3x+y-1)^{-3/2}(1) = -\frac{1}{4}(3x+y-1)^{-3/2}, \quad f_x(1,-1) = -\frac{1}{4}$$

Thus,

$$\begin{aligned} L_{(1,-1)}(x,y) &= f(1,-1) + f_x(1,-1)(x-1) + f_y(1,-1)(y+1) \\ &= 1 + \frac{3}{2}(x-1) + \frac{1}{2}(y+1) \end{aligned}$$

and

$$\begin{aligned} T_2(x,y) &= f(1,-1) + f_x(1,-1)(x-1) + f_y(1,-1)(y+1) \\ &\quad + \frac{1}{2}\left(f_{xx}(1,-1)(x-1)^2 + 2f_{xy}(1,-1)(x-1)(y+1) + f_{yy}(1,-1)(y+1)^2\right) \\ &= 1 + \frac{3}{2}(x-1) + \frac{1}{2}(y+1) + \frac{1}{2}\left(-\frac{9}{4}(x-1)^2 - \frac{6}{4}(x-1)(y+1) - \frac{1}{4}(y+1)^2\right) \end{aligned}$$

At $(0.9,-1)$, a (really good approximation of the) true value of f is $f(0.9,-1) = 0.8366600265$. The linear approximation gives

$$L_{(1,-1)}(0.9,-1) = 1 + \frac{3}{2}(0.9-1) + \frac{1}{2}(-1+1) = 0.85$$

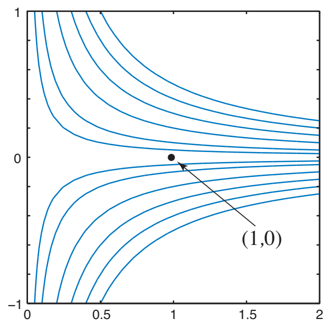
and the degree-2 Taylor approximation is

$$T_2(0.9,-1) = 1 + \frac{3}{2}(-0.1) + \frac{1}{2}(0) + \frac{1}{2}\left(-\frac{9}{4}(-0.1)^2 - \frac{6}{4}(0) - \frac{1}{4}(0)\right) = 0.83875$$

37. We compute: $f = x\sin y$, $f(1,0) = 0$; $f_x = \sin y$, $f_x(1,0) = 0$; $f_y = x\cos y$, $f_y(1,0) = 1$; $f_{xx} = 0$, $f_{xx}(1,0) = 0$; $f_{xy} = f_{yx} = \cos y$, $f_{xy}(1,0) = f_{yx}(1,0) = 1$; and $f_{yy} = -x\sin y$, $f_{yy}(1,0) = 0$; Thus,

$$\begin{aligned} T_2(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)y \\ &\quad + \frac{1}{2}\left(f_{xx}(1,0)(x-1)^2 + 2f_{xy}(1,0)(x-1)y + f_{yy}(1,0)y^2\right) \\ &= 0 + 0 + y + \frac{1}{2}(0 + 2(x-1)y + 0) = y + (x-1)y = xy \end{aligned}$$

The level curves of $T_2(x,y) = xy$ are the hyperbolas $xy = c$ or $y = c/x$ (if $c \neq 0$) and the pair of the x-axis and the y-axis if $c = 0$. The contour diagram of $f(x,y)$ near $(1,0)$ is below (level curves shown are of value (from top to bottom): 0.5, 0.4, 0.3, 0.2, 0.1, 0.05, −0.05, −0.1, −0.2, −0.3, −0.4, and −0.5).



39. To approximate $1.01 \ln 1.08$ we use the degree-2 Taylor polynomial of $f(x,y) = x\ln y$ at the point $(x=1, y=1)$.

We compute: $f = x\ln y$, $f(1,1) = 0$; $f_x = \ln y$, $f_x(1,1) = 0$; $f_y = x/y$, $f_y(1,1) = 1$; $f_{xx} = 0$,

$f_{xx}(1,1) = 0$; $f_{xy} = f_{yx} = 1/y$, $f_{xy}(1,1) = 1$; and $f_{yy} = -x/y^2$, $f_{yy}(1,1) = -1$. Thus,

$$\begin{aligned} T_2(x,y) &= f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) \\ &\quad + \frac{1}{2}\left(f_{xx}(1,1)(x-1)^2 + 2f_{xy}(1,1)(x-1)(y-1) + f_{yy}(1,1)(y-1)^2\right) \\ &= 0 + 0 + (y-1) + \frac{1}{2}\left(0 + 2(x-1)(y-1) - 1(y-1)^2\right) \\ &= y - 1 + (x-1)(y-1) - \frac{1}{2}(y-1)^2 \end{aligned}$$

It follows that

$$\begin{aligned} 1.01\ \ln 1.08 &= f(1.01, 1.08) \approx T_2(1.01, 1.08) \\ &= 1.08 - 1 + (1.01-1)(1.08-1) - \frac{1}{2}(1.08-1)^2 = 0.07760 \end{aligned}$$

A true value (actually a very good approximation) is $f(1.01, 1.08) = 0.07773065155$.

41. To approximate $e^{0.9^2 - 0.05^2}$ we use the degree-2 Taylor polynomial of $f(x,y) = e^{x^2-y^2}$ at the point $(x = 1, y = 0)$.

We compute: $f = e^{x^2-y^2}$, $f(1,0) = e$; $f_x = 2xe^{x^2-y^2}$, $f_x(1,0) = 2e$; $f_y = -2ye^{x^2-y^2}$, $f_y(1,0) = 0$; $f_{xx} = 2e^{x^2-y^2} + 2xe^{x^2-y^2}(2x)$, $f_{xx}(1,0) = 6e$; $f_{xy} = f_{yx} = 2xe^{x^2-y^2}(-2y)$, $f_{xy}(1,0) = 0$; and $f_{yy} = -2e^{x^2-y^2} - 2ye^{x^2-y^2}(-2y)$, $f_{yy}(1,1) = -2e$; Thus,

$$\begin{aligned} T_2(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)y \\ &\quad + \frac{1}{2}\left(f_{xx}(1,0)(x-1)^2 + 2f_{xy}(1,0)(x-1)y + f_{yy}(1,0)y^2\right) \\ &= e + 2e(x-1) + \frac{1}{2}\left(6e(x-1)^2 - 2ey^2\right) \\ &= e[1 + 2(x-1) + 3(x-1)^2 - y^2] \end{aligned}$$

It follows that

$$\begin{aligned} e^{0.9^2 - 0.05^2} &= f(0.9, 0.05) \approx T_2(0.9, 0.05) \\ &= e[1 + 2(0.9-1) + 3(0.9-1)^2 - (0.05)^2] \approx 2.249378213 \end{aligned}$$

A true value (actually a very good approximation) is $f(0.9, 0.05) = 2.242295236$.

43. Think of it this way: imagine that we have two slots ___ ___ and have to place the name of a variable in each slot. Consider two situations: (a) the mixed second partial derivatives are not equal (i.e., Theorem 11 does not hold), and (b) Theorem 11 holds, i.e., the mixed partial derivatives are equal.

Case (a): if a function has 3 variables, then we have 3 options to fill in the first slot, and 3 options to fill in the second slot; thus there are $3^2 = 9$ possible arrangements, i.e., 9 distinct second derivatives (xx, xy, xz, yx, yy, yz, zx, zy, and zz). In the case of n variables, there are n^2 distinct second partial derivatives.

Case (b): of the nine possibilities listed in (a), $xy = yx$, $xz = zx$, and $yz = zy$. Thus, there are 6 distinct second partial derivatives of a function of three variables. We could have calculated this in the following way: of the total of 3^2 possibilities, there are $3^2 - 3$ derivatives that involve two distinct variables. Since the order does not matter, we need to remove half of them: $(3^2 - 3)/2 = 3$. Thus, there is a total of $3^2 - 3 = 6$ derivatives. If a function has n variables, then: the total number of second derivatives is n^2 and the number of derivatives involving the same variables (such as xx, yy, zz and so on) is n. Thus, there are $n^2 - n$ derivatives that involve two distinct variables. We need to remove one half of those, so there is a total of

$$n^2 - \frac{n^2 - n}{2} = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}$$

distinct second-order derivatives.

Section 8 Partial Differential Equations

1. Let $f(x) = x^2 - 1$; then $u(x,t) = f(x-3t) = (x-3t)^2 - 1$, and so u is a travelling wave of speed 3 (units).

From $u_t = 2(x-3t)(-3)$ and $u_x = 2(x-3t)$ we conclude that u is a solution of the partial differential equation $u_t(x,t) = -su_x(x,t)$ with $s = 3$. That's fine, but all this means is that $u(x,t)$ *could* be a travelling wave. Note that we proved that a travelling wave u satisfies $u_t(x,t) = -su_x(x,t)$, but we did not prove it the other way around: namely, we did not show that a solution of $u_t(x,t) = -su_x(x,t)$ is necessarily a travelling wave.

3. This is a straightforward calculation: $f_x = 4x^3 - 12xy^2$, $f_{xx} = 12x^2 - 12y^2$; $f_y = -12x^2y + 4y^3$, and $f_{yy} = -12x^2 + 12y^2$. Clearly, $f_{xx} + f_{yy} = 0$.

5. From $f_x = e^x \sin y$, $f_{xx} = e^x \sin y$, $f_y = e^x \cos y$, and $f_{yy} = -e^x \sin y$ we get $f_{xx} + f_{yy} = 0$.

7. We compute:

$$\begin{aligned}
f_x &= \frac{1}{1+\frac{y^2}{x^2}} y \left(-\frac{1}{x^2}\right) = \frac{-y}{x^2+y^2} \\
f_{xx} &= -y(-1)(x^2+y^2)^{-2}(2x) = \frac{2xy}{x^2+y^2} \\
f_y &= \frac{1}{1+\frac{y^2}{x^2}} \frac{1}{x} = \frac{1}{1+\frac{y^2}{x^2}} \frac{1}{x}\frac{x}{x} = \frac{x}{x^2+y^2} \\
f_{yy} &= x(-1)(x^2+y^2)^{-2}(2y) = -\frac{2xy}{x^2+y^2}
\end{aligned}$$

Clearly, $f_{xx} + f_{yy} = 0$.

9. From $c(x,t) = e^{Ax+Bt}$ we get $c_t(x,t) = Be^{Ax+Bt}$, $c_x(x,t) = Ae^{Ax+Bt}$, and $c_{xx}(x,t) = A^2e^{Ax+Bt}$. Thus

$$c_t(x,t) = Be^{Ax+Bt} = \frac{B}{A^2}A^2e^{Ax+Bt} = \frac{B}{A^2}c_{xx}(x,t)$$

Thus, $c(x,t) = e^{Ax+Bt}$ satisfies the diffusion equation (8.4) with $\sigma = B/A^2$.

11. It is assumed that $u_t = -su_x$. Differentiating with respect to t, we get $u_{tt} = -su_{xt}$, and differentiating with respect to x, we get $u_{tx} = -su_{xx}$. Assuming that the partial derivatives u_{xt} and u_{tx} are continuous, it follows that $u_{xt} = u_{xt}$ (see Theorem 11 in Section 7). Combining the above equations, we get

$$u_{tt} = -su_{xt} = -su_{tx} = -s(-su_{xx}) = s^2u_{xx}$$

Thus, u satisfies the wave equation (8.2) with $a^2 = s^2$.

13. The partial derivatives are computed to be $u_t = 3\sin 2x \cos 3t$, $u_{tt} = -9\sin 2x \sin 3t$, $u_x = 2\cos 2x \sin 3t$, and $u_{xx} = -4\sin 2x \sin 3t$. Thus,

$$u_{tt} = -9\sin 2x \sin 3t = \frac{9}{4}(-4\sin 2x \sin 3t) = \frac{9}{4}u_{xx}$$

That is, u satisfies the wave equation (8.2) with $a^2 = 9/4$.

15. A straightforward differentiation yields

$$\begin{aligned}
u_t &= e^{x-2t}(-2) + 2(x+2t)(2) = -2e^{x-2t} + 4(x+2t) \\
u_{tt} &- 2e^{x-2t}(-2) + 4(2) = 4e^{x-2t} + 8
\end{aligned}$$

$$u_x = e^{x-2t} + 2(x+2t)$$
$$u_{xx} = e^{x-2t} + 2$$

Thus,

$$u_{tt} = 4e^{x-2t} + 8 = 4(e^{x-2t} + 2) = 4u_{xx}$$

Thus, $u(x,t) = e^{x-2t} + (x+2t)^2$ satisfies the wave equation (8.2) with $a^2 = 4$.

17. (a) From $u(x,t) = \sin x \sin 2t$ we get $u_x(x,t) = \cos x \sin 2t$, $u_{xx}(x,t) = -\sin x \sin 2t$, and so

$$u_{xx}(\pi/4, 0.5) = -\sin(\pi/4)\sin 2(0.5) = -\frac{\sqrt{2}}{2}\sin 1 \approx -0.595$$

Thus, when $t = 0.5$, at the location $x = \pi/4$, the string is concave down.

(b) From $u(x,t) = \sin x \sin 2t$ we get $u_t(x,t) = 2\sin x \cos 2t$, $u_{tt}(x,t) = -4\sin x \sin 2t$, and so

$$u_t(\pi/4, 0.5) = 2\sin(\pi/4)\cos 1 = \sqrt{2}\cos 1 \approx 0.764$$
$$u_{tt}(\pi/4, 0.5) = -4\sin(\pi/4)\sin 1 = -2\sqrt{2}\sin 1 \approx -2.380$$

When $t = 0.5$, at the location $x = \pi/4$, the speed of the string is positive, so it moves upward. The fact that u_{tt} is negative means that it is decelerating, i.e., it is slowing down.

Section 9 Directional Derivative and Gradient

1. By the definition of the directional derivative,

$$\begin{aligned} D_{\mathbf{u}}f(2,-3) &= \lim_{h\to 0}\frac{f(2+h(1/\sqrt{5}),-3+h(2/\sqrt{5}))-f(2,-3)}{h} \\ &= \lim_{h\to 0}\frac{(2+h/\sqrt{5})^2-(-3+2h/\sqrt{5})-7}{h} \end{aligned}$$

3. At all points on the contour curve through (a,b) the function f has the same value (equal to $f(a,b)$). Thus, the rate of change of f along this contour curve is zero. Intuitively: the tangent line through (a,b) approximates the contour curve, and thus the rate of change of f along the tangent is approximated by zero. (To make this argument work, we need to assume that f is differentiable.)

5. The maximum value of the directional derivative at $(2,3)$ is $\|\nabla f(2,3)\| = \sqrt{4^2+1^2} = \sqrt{17}$. Since $4 < \sqrt{17}$, there is a direction $\mathbf{u}$ where $D_{\mathbf{u}}f(2,3) = 4$. From $D_{\mathbf{u}}f(2,3) = 4$ we get (keep in mind that $\mathbf{u}$ is a unit vector)

$$\begin{aligned} \|\nabla f(2,3)\|\cdot\|\mathbf{u}\|\cdot\cos\theta &= 4 \\ \sqrt{17}\cos\theta &= 4 \\ \cos\theta &= \frac{4}{\sqrt{17}} \end{aligned}$$

Using a calculator, we find that $\theta \approx 0.25$ (radians). Thus, there are two directions where $D_{\mathbf{u}}f(2,3) = 4$: they make an angle of approximately 0.25 radians with respect to $\nabla f(2,3)$.

7. Think of the curve given by $F(x,y) = 0$ as a level curve of value zero of the function $F(x,y)$. At a point (a,b), the gradient $\nabla F(a,b) = (F_x(a,b), F_y(a,b))$ is perpendicular to the level curve (and, therefore, perpendicular to the tangent line). Thus, any non-zero vector perpendicular to $\nabla F(a,b)$ can serve as a direction vector of the tangent line; take, for instance, $\mathbf{v} = (-F_y(a,b), F_x(a,b))$. The slope of the line whose direction is given by $\mathbf{v}$ is $-F_x(a,b)/F_y(a,b)$. Using the point-slope form, we obtain the equation of the tangent line:

$$y - b = -\frac{F_x(a,b)}{F_y(a,b)}(x-a)$$

9. At B, we have $f(B) = 1$. Moving in the direction of $\mathbf{v} = \mathbf{i}+\mathbf{j}$ we meet the level curve of value 2 at the point $C = (3,3)$; thus, $f(C) = 2$. The distance between B and C is $\sqrt{2}$, and so

$$D_{\mathbf{u}}f(B) \approx \frac{f(C)-f(B)}{\sqrt{2}} = \frac{2-1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

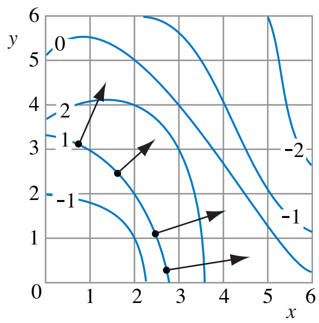
11. At C, we have $f(C) = 2$. Moving in the direction of $\mathbf{v} = -\mathbf{i}+2\mathbf{j}$ we meet the level curve of value 0 at the point $(2,5)$; thus, $f(2,5) = 0$. The distance between C and $(2,5)$ is $\sqrt{1^2+2^2} = \sqrt{5}$, and so

$$D_{\mathbf{u}}f(C) \approx \frac{f(2,5)-f(C)}{\sqrt{5}} = \frac{0-2}{\sqrt{5}} = -\frac{2}{\sqrt{5}}$$

13. We see that $f(E) = -2$. Moving in the direction of $\mathbf{v} = -2\mathbf{i}+\mathbf{j}$ we meet the level curve of value -1 at the point $D = (3.5,5)$; thus, $f(D) = -1$. The distance between E and D is $\sqrt{2^2+1^2} = \sqrt{5}$, and so

$$D_{\mathbf{u}}f(E) \approx \frac{f(D)-f(E)}{\sqrt{5}} = \frac{-1-(-2)}{\sqrt{5}} = \frac{1}{\sqrt{5}}$$

15. See the figure below. Keep in mind two facts: (a) the gradient vector is perpendicular to a contour curve (in this case, the contour curve of value 1) and points in the direction of increasing values of the function (in this case, in the direction toward the contour curve of value 2). (b) The closer (farther apart) the two contour curves are, the larger (smaller) the magnitude of the gradient vector.



17. From $\nabla f(x,y) = (3x^2y + 2y^2, x^3 + 4xy - 3y^2)$ we get $\nabla f(1,3) = (27, -14)$. The unit vector in the direction of $\mathbf{v} = 2\mathbf{i} + \mathbf{j}$ is $\mathbf{u} = \mathbf{v}/\|\mathbf{v}\| = (2\mathbf{i}+\mathbf{j})/\sqrt{5} = (2/\sqrt{5})\mathbf{i} + (1/\sqrt{5})\mathbf{j}$. Thus

$$D_{\mathbf{u}}f(1,3) = \nabla f(1,3)\cdot\mathbf{u} = (27\mathbf{i} - 14\mathbf{j})\cdot\left(\frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}\right) = \frac{54}{\sqrt{5}} - \frac{14}{\sqrt{5}} = \frac{40}{\sqrt{5}} \approx 17.8885$$

19. From

$$\nabla f(x,y) = \left(\frac{1}{2}(x^2+y^2)^{-1/2}(2x), \frac{1}{2}(x^2+y^2)^{-1/2}(2y)\right) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

we get $\nabla f(-2,-2) = (-2/8, -2/8) = (-1/4, -1/4)$. The unit vector in the direction of $\mathbf{v} = -\mathbf{i} + \mathbf{j}$ is $\mathbf{u} = \mathbf{v}/\|\mathbf{v}\| = (-\mathbf{i}+\mathbf{j})/\sqrt{2} = (-1/\sqrt{2})\mathbf{i} + (1/\sqrt{2})\mathbf{j}$. Thus

$$D_{\mathbf{u}}f(-2,-2) = \nabla f(-2,-2)\cdot\mathbf{u} = (-1/4, -1/4)\cdot(-1/\sqrt{2}, 1/\sqrt{2}) = 0$$

21. From

$$\nabla f(x,y) = \left(\ln y^2, x\frac{1}{y^2}(2y) + 2\right) = \left(\ln y^2, \frac{2x}{y} + 2\right)$$

we get $\nabla f(3,5) = (\ln 25, 6/5 + 2) = (\ln 25, 16/5)$. The unit vector in the direction of $\mathbf{v} = 8\mathbf{i} + 6\mathbf{j}$ is $\mathbf{u} = \mathbf{v}/\|\mathbf{v}\| = (8\mathbf{i}+6\mathbf{j})/10 = (4/5)\mathbf{i} + (3/5)\mathbf{j}$. (Note that $\|\mathbf{v}\| = \sqrt{8^2+6^2} = \sqrt{100} = 10$.) Thus

$$D_{\mathbf{u}}f(3,5) = \nabla f(3,5)\cdot\mathbf{u} = (\ln 25, 16/5)\cdot(4/5, 3/5) = \frac{4}{5}\ln 25 + \frac{3}{5}\frac{16}{5} \approx 4.4951$$

23. From $\nabla f(x,y) = 6y\mathbf{i} + 6x\mathbf{j}$ we get $\nabla f(1,2) = 12\mathbf{i} + 6\mathbf{j}$ and $\|\nabla f(1,2)\| = \|12\mathbf{i} + 6\mathbf{j}\| = 6\|2\mathbf{i} + \mathbf{i}\| = 6\sqrt{5}$. Denoting by $\mathbf{u}$ the unit vector whose direction makes an angle θ with respect to $\nabla f(1,2)$, we get

$$\begin{aligned} D_{\mathbf{u}}f(1,2) &< 2 \\ \|\nabla f(1,2)\|\cdot\|\mathbf{u}\|\cdot\cos\theta &< 2 \\ 6\sqrt{5}\cos\theta &< 2 \\ \cos\theta &< \frac{2}{6\sqrt{5}} \end{aligned}$$

Using a calculator, we find that $\cos\theta = 2/6\sqrt{5}$ when $\theta \approx 1.421$ radians.

So, in the directions whose angle with respect to $\nabla f(1,2)$ is larger than 1.421 radians (and smaller than or equal to π radians) the directional derivative of f is smaller than 2.

25. From

$$\nabla f(x,y) = (2xy^{-2}, x^2(-2)y^{-3}) = \left(\frac{2x}{y^2}, -\frac{2x^2}{y^3}\right)$$

we get $\nabla f(2,-1) = (4,8)$ and $\|\nabla f(2,-1)\| = \|(4,8)\| = 4\|(1,2)\| = 4\sqrt{5}$. The maximum rate of change of f at $(2,-1)$ is $4\sqrt{5}$; it occurs in the direction of the gradient $\nabla f(2,-1) = (4,8)$; the corresponding unit direction is $u = \nabla f(2,-1)/\|\nabla f(2,-1)\| = (4,8)/4\sqrt{5} = (1/\sqrt{5}, 2/\sqrt{5})$.

27. From

$$\nabla f(x,y) = \left(y\frac{1}{2}x^{-1/2}, \sqrt{x}+3y^2\right) = \left(\frac{y}{2\sqrt{x}}, \sqrt{x}+3y^2\right)$$

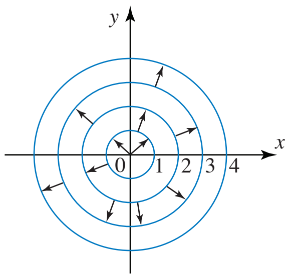
we get $\nabla f(1,2) = (1,13)$ and $\|\nabla f(1,2))\| = \|(1,13)\| = \sqrt{1^2+13^2} = \sqrt{170}$. The maximum rate of change of f at $(1,2)$ is $\sqrt{170}$; it occurs in the direction of the gradient $\nabla f(1,2) = (1,13)$; the corresponding unit direction is $u = \nabla f(1,2)/\|\nabla f(1,2)\| = (1/\sqrt{170}, 13/\sqrt{170})$.

29. Starting at A, where $f(A) = -1$, and moving in the direction perpendicular to the level curve of value -1 toward the larger values of f, we meet the level curve of value 2 after covering the distance of about 1/2 units. At B, the value of f is 1. Moving in the direction perpendicular to the level curve of value 1 toward the level curves of larger values, we meet the level curve of value 2 after walking the distance a bit larger than one unit. So, at A the function f increases by about 2 units over the distance of 1/2 units. At B, f increases by about 1 unit over the distance larger than 1 unit. Thus, $\|\nabla f(A)\| > \|\nabla f(B)\|$.

31. The function $f(x,y)$ measures the distance between a point with coordinates (x,y) and the origin. The level curves of f are given by $\sqrt{x^2+y^2} = c$, i.e., $x^2+y^2 = c^2$; thus, they are concentric circles centred at the origin (makes sense, since f is the distance from the origin). The gradient of f is

$$\nabla f(x,y) = \left(\frac{1}{2}(x^2+y^2)^{-1/2}(2x), \frac{1}{2}(x^2+y^2)^{-1/2}(2y)\right) = \frac{1}{\sqrt{x^2+y^2}}(x,y)$$

Note that $\nabla f(x,y)$ is a unit vector, parallel to the position vector of a point (x,y). Thus: all gradient vectors are of the same length (since the distance changes in the same way at all points) and point radially away from the origin; see the figure below.



33. It is given that $D_{\mathbf{u}}f(1,3) = 10$, where $\mathbf{u}$ is the unit vector from $(1,3)$ to $(2,2)$. We compute $\mathbf{u} = (1,-1)/\sqrt{2} = (1/\sqrt{2}, -1/\sqrt{2})$. The gradient of f at $(1,3)$ is $\nabla f(1,3) = (f_x(1,3), f_y(1,3))$. Now

$$\begin{aligned} D_{\mathbf{u}}f(1,3) &= 10 \\ \nabla f(1,3)\cdot\mathbf{u} &= 10 \\ f_x(1,3)\frac{1}{\sqrt{2}} - f_y(1,3)\frac{1}{\sqrt{2}} &= 10 \\ f_x(1,3) - f_y(1,3) &= 10\sqrt{2} \end{aligned}$$

It is also given that $D_{\mathbf{u}}f(1,3) = 6$, where $\mathbf{u}$ is the unit vector from $(1,3)$ to $(2,3)$. Since $\mathbf{u} = (1,0) = \mathbf{i}$, $D_{\mathbf{u}}f(1,3) = f_x(1,3)$, and so $f_x(1,3) = 6$. From $f_x(1,3) - f_y(1,3) = 10\sqrt{2}$ it follows that $f_y(1,3) = 6 - 10\sqrt{2}$. Thus, $\nabla f(1,3) = (6, 6-10\sqrt{2})$.

35. (a) The gradient of P is

$$\nabla P(x,y) = 40e^{-2x^2-y^2}(-4x)\mathbf{i} + 40e^{-2x^2-y^2}(-2y)\mathbf{j} = -80e^{-2x^2-y^2}(2x\mathbf{i} + y\mathbf{j})$$

and $\nabla P(1,0) = -80e^{-2}(2\mathbf{i}) = -160e^{-2}\mathbf{i}$. The unit vector in the direction of $\mathbf{i}-\mathbf{j}$ is $(1/\sqrt{2})\mathbf{i}-(1/\sqrt{2})\mathbf{j}$. Thus, the rate of change of the pressure at the point $(1,0)$ in the direction $\mathbf{i}-\mathbf{j}$ is

$$D_{\mathbf{u}}P(1,0) = \nabla P(1,0)\cdot\mathbf{u} = (-160e^{-2}\mathbf{i})\cdot\left(\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}\right) = -\frac{160e^{-2}}{\sqrt{2}} \approx 15.311$$

(b) The pressure increases most rapidly in the direction of $\nabla P(1,0)$, which is $-\mathbf{i}$. It decreases most rapidly in the opposite direction, which is $\mathbf{i}$.

(c) The maximum rate of increase in P at $(1,0)$ is

$$\|\nabla P(1,0)\| = \|-160e^{-2}\mathbf{i}\| = 160e^{-2} \approx 21.654$$

(d) The rate of change of pressure is zero in the directions perpendicular to the gradient, i.e., in the directions of the vectors $\mathbf{j}$ and $-\mathbf{j}$.

37. (a) Note that $T(x,y) > 0$ for all (x,y). As well, since $x^2+y^2 \geq 0$, we get $x^2+y^2+4 \geq 4$ and

$$T(x,y) = \frac{120}{x^2+y^2+4} \leq \frac{120}{4} = 30$$

Thus, $0 < T(x,y) \leq 30$ for all (x,y) in $\mathbb{R}^2$. The level curve of T of value c is given by

$$\frac{120}{x^2+y^2+4} = c$$
$$x^2+y^2+4 = \frac{120}{c}$$
$$x^2+y^2 = \frac{120}{c} - 4$$

The contour diagram consists of concentric circles (centred at the origin); a level curve of value c has the radius of $\sqrt{120/c-4}$, if $0 < c \leq 30$ (otherwise, a level curve is an empty set). See the figure below, left (the level curve of value $c=30$ has the radius 0 (i.e., is a point); when $c=24$ the radius is 1, when $c=20$ the radius is $\sqrt{2}$, when $c=10$ the radius is $\sqrt{8}$, and when $c=1$ the radius is $\sqrt{116}$).

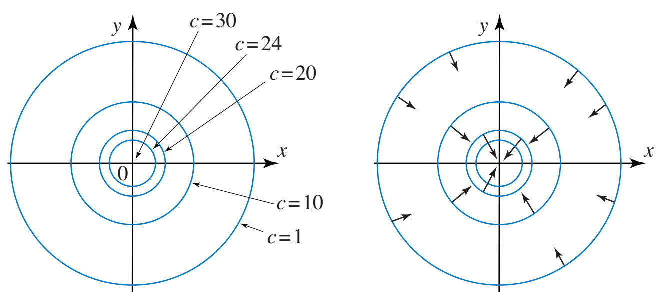
(b) We compute

$$\nabla T(x,y) = \big(120(-1)(x^2+y^2+4)^{-2}(2x), 120(-1)(x^2+y^2+4)^{-2}(2y)\big) = -\frac{240}{(x^2+y^2+4)^2}(x,y)$$

Recall that $\mathbf{v} = (x,y)$ is the position vector of a point (x,y) (i.e., the vector from $(0,0)$ to (x,y)). The gradient $\nabla T(x,y)$ points in the opposite direction, toward the origin. All gradient vectors whose tails are on the same circle (say, $x^2+y^2=r^2$) have the same magnitude, equal to

$$\|\nabla T\| = \frac{240}{r^2+4}r = \frac{240r}{r^2+4}$$

Thus, when $r=1$ their length is $240/5=48$, when $r=\sqrt{2}$ their length is $40\sqrt{2}\approx 56.6$ when $r=\sqrt{8}$ their length is $20\sqrt{8}\approx 56.6$, and when $r=\sqrt{116}$ their length is approximately 21.5; see the figure below, right (we did not show the true size of the gradient vectors.)



(c) Since gradient vectors point toward the origin, the warmest point is the origin. There, $T(0,0) = 120/4 = 30$.

(d) In (a) we showed that $0 < T(x,y) \leq 30$, i.e., T cannot be larger that 30. Thus, $T(0,0) = 120/4 = 30$ is the maximum.

39. Let $f(x,y) = e^y - 2xy^3 + 4x$; we think of the given curve as a contour curve of f of value 5. The gradient of f is $\nabla f(x,y) = (-2y^3 + 4, e^y - 6xy^2)$. Since the vector $\nabla f(1,0) = (4,1)$ is perpendicular to the contour curve (i.e., to the given curve) at the point $(1,0)$, we can take any vector perpendicular to it as a direction vector of the tangent line. Take, for instance, $\mathbf{v} = (-1,4)$. The equation of the tangent line (in parametric form) is

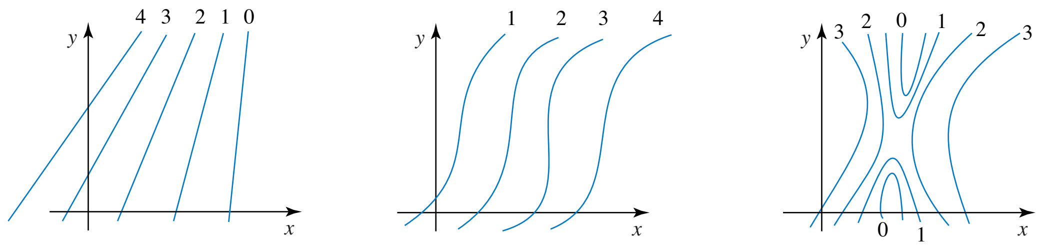
$$\begin{aligned} x &= 1 + t(-1) = 1 - t \\ y &= 0 + t(4) = 4t \end{aligned}$$

where $t \in \mathbb{R}$. Substituting $t = y/4$ from the second equation into $x = 1 - t$, we get $x = 1 - y/4$, $4x = 4 - y$ and $y = -4x + 4$.

Alternatively: the slope of the line whose direction vector is $(-1,4)$ is $4/(-1) = -4$. Using the point-slope form, we obtain $y - 0 = -4(x-1)$, i.e., $y = -4x + 4$.

Section 10 Extreme Values

1. A linear function $f(x, y) = ax + by + c$ (where at least one of a or b is non-zero) has no extreme values. Thus, we could draw a contour diagram of a linear function. Few more examples are provided in the pictures below.

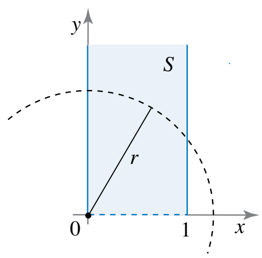


3. Calculating $f_x = 4x^3 - 4y^2$ and $f_y = -8xy$, we see that $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$; so $(0, 0)$ is indeed a critical point of f. To show that it is a saddle point, we will show that it is neither a local minimum point nor a local maximum point.

When $y = 0$, then $f_1(x) = f(x, 0) = x^4$. Thus, the values of f along the x-axis suggest that f has a local minimum at $x = 0$. Along the curve $y^2 = x^3$, we get $f_2(x) = x^4 - 4x(x^3) = -3x^4$. This suggests that f has a local maximum at $x = 0$ (i.e., at $(0, 0)$). We are done.

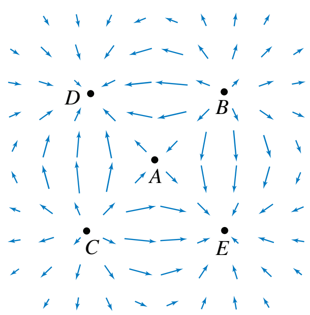
Alternatively, consider the values of f along $y = x$; we get $f_3(x) = x^4 - 4x^3$. Using one-variable calculus, we see that $f_3'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$. For small values of x on either side of 0, f_3' is negative. Thus, f_3 is decreasing both before and after 0, and so 0 is neither a minimum nor a maximum of f_3. Since the graph of $f(x, y)$ contains the curve f_3, we conclude that $f(0, 0)$ is neither minimum nor maximum of f.

5. Pick an open disk of radius r centred at the origin. No matter what r we take, the vertical strip will contain points whose distance from the origin is larger than r, i.e., which not belong to the open disk. See the figure below.



7. The gradient of f is non-zero at all points in $\mathbb{R}^2$; thus, f has no critical points.

9. There are five critical points, see the figure below.



Consider the point A: along one direction, the gradient points towards A, and along the perpendicular direction, it points away from A. Thus, A is a saddle point.

The gradient vectors around B point away from B, indicating that f increases in the directions away from B; thus, B is a local minimum. The same argument proves that C is a local minimum. The gradient vectors point toward D and E, indicating that f has a local maximum values at these two points.

11. The partial derivatives of z are: $z_x = 2x$, $z_y = -2y$, $z_{xx} = 2$, $z_{xy} = 0$, and $z_{yy} = -2$. Thus, $D(x,y) = z_{xx}z_{yy} - z_{xy}^2 = -4$; from $D(0,0) = -4 < 0$ we conclude that z has a saddle point at $(0,0)$.

13. To find the critical points, we solve the system $f_x = 2x - 5y = 0$ and $f_y = -5x - 2y = 0$. Substituting $y = 2x/5$ (which we obtained from the first equation) into the second equation, we get $-5x - 4x/5 = 0$ and $-29x/5 = 0$, i.e., $x = 0$. It follows that $y = 0$, and so $(0,0)$ is the only critical point. (Note that f_x and f_y are defined at all $(x,y) \in \mathbb{R}^2$.)

[Alternatively, for those familiar with basics of linear systems: the determinant of the homogeneous system $f_x = 2x - 5y = 0$ and $f_y = -5x - 2y = 0$ is $-29 \neq 0$, and so the system has only the trivial solution $x = 0$ and $y = 0$.]

From $f_{xx} = 2$, $f_{xy} = -5$ and $f_{yy} = -2$ we obtain $D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = -4 - (-5)^2 = -29$. Since $D(0,0) = -29 < 0$, we conclude that $(0,0)$ is a saddle point.

15. To find critical points, we solve the system

$$f_x = ye^{-x^2-y^2} + xye^{-x^2-y^2}(-2x) = ye^{-x^2-y^2}(1-2x^2) = 0$$
$$f_y = xe^{-x^2-y^2} + xye^{-x^2-y^2}(-2y) = xe^{-x^2-y^2}(1-2y^2) = 0$$

i.e.,

$$y(1-2x^2) = 0$$
$$x(1-2y^2) = 0$$

The first equation implies that $y = 0$ or $1 - 2x^2 = 0$, i.e., $x^2 = 1/2$ and $x = \pm 1/\sqrt{2}$. Substituting $y = 0$ into the second equation we get $x = 0$. Thus, $(0,0)$ is a critical point.

Substituting $x = \pm 1/\sqrt{2}$ into the second equation we get $(\pm 1/\sqrt{2})(1-2y^2) = 0$, which implies that $1 - 2y^2 = 0$ and $y = \pm 1/\sqrt{2}$. We obtained four more critical points: $(1/\sqrt{2}, 1/\sqrt{2})$, $(-1/\sqrt{2}, 1/\sqrt{2})$, $(1/\sqrt{2}, -1/\sqrt{2})$, and $(-1/\sqrt{2}, -1/\sqrt{2})$. There are no more critical points, as f_x and f_y are defined for all $(x,y) \in \mathbb{R}^2$.

Now the second partials:

$$\begin{aligned} f_{xx} &= ye^{-x^2-y^2}(-2x)(1-2x^2) + ye^{-x^2-y^2}(-4x) \\ &= -2xye^{-x^2-y^2}(1-2x^2+2) = -2xye^{-x^2-y^2}(3-2x^2) \\ f_{xy} &= e^{-x^2-y^2}(1-2x^2) + ye^{-x^2-y^2}(-2y)(1-2x^2) \\ &= e^{-x^2-y^2}(1-2x^2)(1-2y^2) \\ f_{yy} &= xe^{-x^2-y^2}(-2y)(1-2y^2) + xe^{-x^2-y^2}(-4y) \\ &= -2xye^{-x^2-y^2}(1-2y^2+2) = -2xye^{-x^2-y^2}(3-2y^2) \end{aligned}$$

and therefore

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 4x^2y^2(3-2x^2)(3-2y^2)e^{-2x^2-2y^2} - e^{-2x^2-2y^2}(1-2x^2)^2(1-2y^2)^2$$

We now test each critical point. Since $D(0,0) = 0 - 1 = -1 < 0$, it follows that $(0,0)$ is a saddle point. For the remaining four critical points,

$$D(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}) = 4\frac{1}{2}\frac{1}{2}(2)(2)e^{-2} - e^{-2}(0) = 4e^{-2} > 0$$

Note that

$$f_{xx} = -xy[2e^{-x^2-y^2}(3-2x^2)] = -xy \cdot [\text{positive quantity when } x = \pm 1/\sqrt{2}]$$

Thus,

$$f_{xx}(1/\sqrt{2}, 1/\sqrt{2}) < 0; \text{ so } f \text{ has a relative maximum at } (1/\sqrt{2}, 1/\sqrt{2})$$
$$f_{xx}(-1/\sqrt{2}, -1/\sqrt{2}) < 0; \text{ so } f \text{ has a relative maximum at } (-1/\sqrt{2}, -1/\sqrt{2})$$
$$f_{xx}(-1/\sqrt{2}, 1/\sqrt{2}) > 0; \text{ so } f \text{ has a relative minimum at } (-1/\sqrt{2}, 1/\sqrt{2})$$
$$f_{xx}(1/\sqrt{2}, -1/\sqrt{2}) > 0; \text{ so } f \text{ has a relative minimum at } (1/\sqrt{2}, -1/\sqrt{2})$$

17. To find critical points, we solve the system

$$f_x = (3x^2 - 1)e^{-y^2} = 0$$
$$f_y = (x^3 - x)e^{-y^2}(-2y) = 0$$

From the first equation we get $3x^2 - 1 = 0$; thus, $x^2 = 1/3$ and $x = \pm 1/\sqrt{3}$. Note that the factor $x^3 - x$ in the second equation is not zero when $x = \pm 1/\sqrt{3}$. Thus, $f_y = 0$ implies that $y = 0$, and so there are two critical points: $(1/\sqrt{3}, 0)$ and $(-1/\sqrt{3}, 0)$. (There are no other critical points, since f_x and f_y are defined for all $(x, y) \in \mathbb{R}^2$.)

The second partial derivatives of f are

$$f_{xx} = 6xe^{-y^2}$$
$$f_{xy} = (3x^2 - 1)e^{-y^2}(-2y) = -2y(3x^2 - 1)e^{-y^2}$$
$$f_{yy} = (x^3 - x)[e^{-y^2}(-2y)(-2y) + e^{-y^2}(-2)] = (x^3 - x)e^{-y^2}(-2)(-2y^2 + 1)$$

Thus,

$$\begin{aligned} D(x, y) &= f_{xx}f_{yy} - f_{xy}^2 \\ &= 6xe^{-y^2}(x^3 - x)e^{-y^2}(-2)(-2y^2 + 1)e^{-y^2} - \left(-2y(3x^2 - 1)e^{-y^2}\right)^2 \\ &= -12(x^4 - x^2)(-2y^2 + 1)e^{-2y^2} - 4y^2(3x^2 - 1)^2 e^{-2y^2} \end{aligned}$$

From

$$D(\pm 1/\sqrt{3}, 0) = -12\left(\frac{1}{9} - \frac{1}{3}\right) \cdot 1 \cdot 1 - 0 > 0$$

$f_{xx}(1/\sqrt{3}, 0) = 6(1/\sqrt{3}) > 0$, and $f_{xx}(-1/\sqrt{3}, 0) = 6(-1/\sqrt{3}) < 0$ we conclude that f has a local maximum at $(-1/\sqrt{3})$ and a local minimum at $(1/\sqrt{3})$.

19. The system of equations for the critical points is

$$f_x = \cos y = 0$$
$$f_y = -x \sin y = 0$$

The first equation implies that $y = \pi/2 + \pi k$, where k is an integer. At all these values $\sin y \neq 0$, and thus from the second equation it follows that $x = 0$. The critical points are $(0, \pi/2 + \pi k)$. (There are no other critical points since f_x and f_y are defined for all $(x, y) \in \mathbb{R}^2$.) From $f_{xx} = 0$, $f_{xy} = -\sin y$, and $f_{yy} = -x \cos y$ we compute $D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = -\sin^2 y$. The fact that $D(0, \pi/2 + \pi k) = -\sin^2(\pi/2 + \pi k) = -(\pm 1)^2 = -1 < 0$ means that all critical points are saddle points.

21. Writing $f(x, y) = x + y + x^{-1}y^{-1}$, we get

$$f_x = 1 + (-1)x^{-2}y^{-1} = 1 - \frac{1}{x^2 y} = 0, \quad \text{and} \quad x^2 y = 1$$
$$f_y = 1 + (-1)y^{-2}x^{-1} = 1 - \frac{1}{xy^2} = 0, \quad \text{and} \quad xy^2 = 1$$

Substituting $y = 1/x^2$ (which we obtained from the first equation) into the second equation, we get

$$x\left(\frac{1}{x^2}\right)^2 = 1$$

and so $1/x^3 = 1$, $x^3 = 1$ and $x = 1$. From $y = 1/x^2$ we compute $y = 1$, and so $(1,1)$ is a critical point.

(There are no other critical points, as the first partial derivatives are defined at all (x, y) where f is defined; i.e., there are no points in the domain of f where one or both partial derivatives do not exist.)

We compute $f_{xx} = -(-2)x^{-3}y^{-1} = 2/x^3y$, $f_{xy} = -x^{-2}(-1)y^{-2} = 1/x^2y^2$, and (by symmetry) $f_{yy} = 2/xy^3$. It follows that

$$\begin{aligned} D(x,y) &= f_{xx}f_{yy} - f_{xy}^2 \\ &= \frac{2}{x^3y}\,\frac{2}{xy^3} - \frac{1}{(x^2y^2)^2} = \frac{4}{x^4y^4} - \frac{1}{x^4y^4} = \frac{3}{x^4y^4} \end{aligned}$$

Since $D(1,1) = 3 > 0$ and $f_{xx}(1,1) = 2 > 0$ we conclude that f has a relative minimum at $(1,1)$.

23. To avoid working with the square root, we minimize the distance squared, $f(x, y) = x^2 + y^2$. For the points on the curve, $xy = 1$ and $y = 1/x$. Thus, we are asked to minimize the function (of one variable) $f(x) = x^2 + 1/x^2$. (An alternative way of solving this question is to use the Lagrange multipliers method, which is covered in the next section. Actually this question is out of place here.)

From $f'(x) = 2x - 2/x^3 = 0$ we get $2x^4 - 2 = 0$, $x^4 - 1 = 0$ and $x^4 = 1$. Thus, there are two critical numbers, $x = \pm 1$. (Note that $f'(x)$ is defined at all points in the domain of $f(x)$.) Since $f''(x) = 2 + 6/x^4$ and $f''(\pm 1) = 8 > 0$, both critical points represent local minimum of f. The minimum value of f is $f(\pm 1) = 2$, and so the minimum distance is $\sqrt{2}$. The points $(1,1)$ and $(-1,-1)$ on $xy = 1$ are the two points that are closest to the origin.

25. (a) It is assumed that the ecosystem consists of the three species A, B and C only. The numbers a, b and c represent the percentages of the three species in the total population, and thus must add up to 1. From $a+b+c = 1$ we get $c = 1-a-b$ and thus $H(a, b) = -a\ln a - b\ln b - (1-a-b)\ln(1-a-b)$.

(b) We compute:

$$\begin{aligned} H_a(a,b) &= -\ln a - a\cdot\frac{1}{a} - (-1)\ln(1-a-b) - (1-a-b)\cdot\frac{1}{1-a-b}(-1) \\ &= -\ln a + \ln(1-a-b) \end{aligned}$$

by symmetry, $H_b(a, b) = -\ln b + \ln(1 - a - b)$. To find critical points, we solve

$$\begin{aligned} H_a(a,b) &= -\ln a + \ln(1-a-b) = 0 \\ H_b(a,b) &= -\ln b + \ln(1-a-b) = 0 \end{aligned}$$

which simplifies to $a = 1 - a - b$ and $b = 1 - a - b$. Thus $a = b$; using the first equation, we get $a = 1 - 2a$, $3a = 1$ and $a = 1/3$. So $(1/3, 1/3)$ is a critical point of $H(a, b)$.

The second partials are

$$\begin{aligned} H_{aa} &= -\frac{1}{a} + \frac{1}{1-a-b}(-1) = -\left(\frac{1}{a} + \frac{1}{1-a-b}\right) \\ H_{ab} &= \frac{1}{1-a-b}(-1) = -\frac{1}{1-a-b} \\ H_{bb} &= -\frac{1}{b} + \frac{1}{1-a-b}(-1) = -\left(\frac{1}{b} + \frac{1}{1-a-b}\right) \end{aligned}$$

and thus

$$\begin{aligned} D(a,b) &= H_{aa}H_{bb} - H_{ab}^2 \\ &= \left(\frac{1}{a} + \frac{1}{1-a-b}\right)\left(\frac{1}{b} + \frac{1}{1-a-b}\right) - \frac{1}{(1-a-b)^2} \end{aligned}$$

It follows that $D(1/3, 1/3) = (6)(6) - 9 = 27 > 0$ and $H_{aa}(1/3, 1/3) = -(6) < 0$, so $(a = 1/3, b = 1/3)$ is indeed a maximum. From $c = 1 - a - b$ we get $c = 1/3$. The Shannon index achieves its largest value when the three species are represented in equal proportion in the total population.

27. Because it is a polynomial, f is continuous on S; the set S is closed and bounded. From $f_x = y - 3 = 0$ and $f_y = x + 1 = 0$ it follows that f has only one critical point, $(-1, 3)$. This point belongs to S, and $f(-1, 3) = (-1)(3) - 3(-1) + 3 = 3$.

The boundary of S consists of four line segments.

Along the segment $y = 0$, $-2 \le x \le 1$, the function f is equal to $f(x, 0) = -3x$. This is a decreasing function, so its extremes occur at the boundary points of the segment $-2 \le x \le 1$: $f(-2, 0) = 6$ and $f(1, 0) = -3$.

Along the segment $x = 1$, $0 \le y \le 3$, f is equal to $f(1, y) = 2y - 3$. This is an increasing function, and its extremes occur at the endpoints: $f(1, 0) = -3$ and $f(1, 3) = 3$.

Along $y = 3$, $-2 \le x \le 1$, the function f is equal to $f(x, 3) = 3$. Thus, $f(x, 3) = 3$ for all $x \in [-2, 1]$.

Along $x = -2$, $0 \le y \le 3$, f is equal to $f(-2, y) = -2y + 6 + y = -y + 6$. Since this is a decreasing function, its extreme values are $f(-2, 0) = 6$ and $f(-2, 3) = 3$.

It follows that the absolute maximum of f on S is $f(-2, 0) = 6$ and the absolute minimum is $f(1, 0) = -3$.

29. Note that f is continuous on S (as a product of a polynomial and an exponential function), and S is closed and bounded. The system of equations $f_x = e^y = 0$ and $f_y = xe^y = 0$ has no solutions, as no point (x, y) satisfies both equations.

The boundary of S consists of four line segments.

Along the segment $y = -3$, $-1 \le x \le 1$, the function f is equal to $f(x, -3) = xe^{-3}$. This is an increasing function of x, and so its extremes occur at the endpoints of the interval $[-1, 1]$ for x: $f(-1, -3) = -e^{-3}$ and $f(1, -3) = e^{-3}$.

Along the segment $x = 1$, $-3 \le y \le 3$, the function f is equal to $f(1, y) = e^y$. This is an increasing function of y, and so its extremes occur at the endpoints of the interval $[-3, 3]$ for y: $f(1, -3) = e^{-3}$ and $f(1, 3) = e^3$.

Along the segment $y = 3$, $-1 \le x \le 1$, the function f is equal to $f(x, 3) = xe^3$. This is an increasing function of x, and so its extremes occur at the endpoints of the interval $[-1, 1]$ for x: $f(-1, 3) = -e^3$ and $f(1, 3) = e^3$.

Along the segment $x = -1$, $-3 \le y \le 3$, the function f is equal to $f(-1, y) = -e^y$. This is a decreasing function of y, and so its extremes occur at the endpoints of the interval $[-3, 3]$ for y: $f(-1, -3) = -e^{-3}$ and $f(-1, 3) = -e^3$.

We conclude that the absolute maximum of f on S is $f(1, 3) = e^3$ and the absolute minimum is $f(-1, 3) = -e^3$.

31. Note that f is continuous on S (as the composition of a logarithm function with a polynomial which is positive for all $(x, y) \in \mathbb{R}^2$); as well, the set S is closed and bounded. The system of equations

$$f_x(x, y) = \frac{2x}{x^2 + y^2 + 1} = 0 \quad \text{and} \quad f_y(x, y) = \frac{2y}{x^2 + y^2 + 1} = 0$$

has one solution, $(x = 0, y = 0)$. Thus, f has only one critical point; the value of f is $f(0, 0) = \ln 1 = 0$.

The boundary of S consists of four line segments.

Along the segment $y = 0$, $0 \le x \le 1$, the function f is equal to $f(x, 0) = \ln(x^2 + 1)$. This is an increasing function if $x \ge 0$, and so its extremes occur at the endpoints of the interval $[0, 1]$ for x: $f(0, 0) = \ln 1 = 0$ and $f(1, 0) = \ln 2$.

Along the segment $x = 1$, $0 \le y \le 1$, the function f is equal to $f(1, y) = \ln(y^2 + 2)$. This is an increasing function of y (if $y \ge 0$, as it is here) and so its extremes occur at the endpoints of the interval $[0, 1]$ for y: $f(1, 0) = \ln 2$ and $f(1, 1) = \ln 3$.

Along the segment $y = 1$, $0 \le x \le 1$, the function f is equal to $f(x, 1) = \ln(x^2 + 2)$. This is the same situation (with x replacing y) as the previous line segment. Thus, the extremes are at $f(0, 1) = \ln 2$ and $f(1, 1) = \ln 3$.

Along the segment $x = 0$, $0 \le y \le 1$, the function f is equal to $f(0, y) = \ln(y^2 + 1)$. This is the same situation (with y replacing x) as the first line segment we analyzed. The extremes occur at the endpoints of the interval $[0,1]$ for y: $f(0,0)=0$ and $f(0,1)=\ln 2$.

We conclude that the absolute maximum of f on S is $f(1,1)=\ln 3$ and the absolute minimum is $f(0,0)=0$.

33. Note that T is continuous on S (as the product of a polynomial and an exponential function); as well, the set S is closed and bounded. The system of equations

$$T_x(x,y)=4xe^y=0 \quad \text{and} \quad T_y(x,y)=2x^2e^y=0$$

is satisfied for $x=0$, and for any value y in $[0.1]$. Thus, all points $(0,y)$, $0\le y\le 1$ are critical points for T; at all these points, $T(0,y)=0$. To avoid analyzing f along the four boundary segments, we argue as follows: $T(x,y)=2x^2e^y$ is the product of $2x^2$ and e^y. The largest value of $2x^2$ on $[0,1]$ is equal to 2 and occurs at $x=1$; the largest value of e^y on $[0,1]$ is e, and occurs at $y=1$. Thus, the largest value of T is $T(x=1,y=1)=2e$. The smallest value of T is 0, and it occurs at all points $(0,y)$, where $0\le y\le 1$.

For completeness, we analyze the values of f on the four boundary line segments of S.

Along the segment $y=0$, $0\le x\le 1$, the function f is equal to $f(x,0)=2x^2$. This is an increasing function, and so its extremes occur at the endpoints of the interval $[0,1]$ for x: $f(0,0)=0$ and $f(1,0)=2$.

Along the segment $x=1$, $0\le y\le 1$, the function f is equal to $f(1,y)=2e^y$. This is an increasing function of y, and so its extremes occur at the endpoints of the interval $[0,1]$ for y: $f(1,0)=2$ and $f(1,1)=2e$.

Along the segment $y=1$, $0\le x\le 1$, the function f is equal to $f(x,1)=2ex^2$. The extremes are at $f(0,1)=0$ and $f(1,1)=2e$.

Along the segment $x=0$, $0\le y\le 1$, the function f is equal to $f(0,y)=0$.

We conclude that the absolute maximum of T on S is $f(1,1)=2e$ and the absolute minimum is 0. Thus, the warmest point is $(1,1)$, and the coldest points lie on the segment that bounds S from the left: $\{(0,y) \mid 0\le y\le 1\}$.

35. (a) From $(f_1)_x=4x^3=0$ and $(f_1)_y=4y^3=0$ it follows that $(0,0)$ is a critical point of f_1. Likewise, from $(f_2)_x=-4x^3=0$ and $(f_2)_y=4y^3=0$ we conclude that $(0,0)$ is a critical point of f_2, and from $(f_3)_x=8x^3=0$ and $(f_3)_y=-4y^3=0$ it follows that $(0,0)$ is a critical point of f_3.

(b) We compute $(f_1)_{xx}=12x^2$, $(f_1)_{xy}=0$, $(f_1)_{yy}=12y^2$; thus $D=144x^2y^2$ and $D(0,0)=0$. For f_2, $(f_2)_{xx}=-12x^2$, $(f_2)_{xy}=0$, $(f_2)_{yy}=12y^2$; thus $D=-144x^2y^2$ and $D(0,0)=0$. For f_3, $(f_3)_{xx}=24x^2$, $(f_3)_{xy}=0$, $(f_3)_{yy}=-12y^2$; thus $D=-288x^2y^2$ and $D(0,0)=0$.

(c) Since $f_1(x,y)=x^4+y^4\ge 0$ for all (x,y), it follows that $f_1(0,0)=0$ is a local minimum.

Substituting $x=0$ into f_2, we get $f_2(0,y)=14+y^4$; $f_2(0,y)$ has a minimum at $y=0$; when $y=0$, we get $f_2(x,0)=14-x^4$; $f_2(x,0)$ has a maximum at $x=0$. Thus, $f_2(x,y)$ has a saddle point at $(0,0)$.

When $x=0$, the $f_3(0,y)=-y^4$ has a maximum at $y=0$. When $y=0$, the $f_3(x,0)=2x^4$ has a minimum at $x=0$. Thus, $f_3(x,y)$ has a saddle point at $(0,0)$.

37. (a) Since $-2<x<2$ and $-2<y<2$, it follows that $f_1(x,y)=\sqrt{x^2+y^2}<\sqrt{2^2+2^2}=\sqrt{8}$; however, there is no point (x,y) in S where $f_1(x,y)=\sqrt{8}$. Thus, f does not have an absolute maximum.

Alternatively, think of f as the distance from a point (x,y) in S to the origin. The distance from $(2,2)$ to the origin is $\sqrt{2^2+2^2}=\sqrt{8}$. Using points in S, we can get as close to $(2,2)$ as needed (thus making the values of f close to $\sqrt{8}$). However, since $(2,2)$ is not in S, we cannot make f equal to $\sqrt{8}$.

The fact that f does not have an absolute maximum does not violate the conclusion of Theorem 21, since the theorem does not apply (the assumption on the closeness of S is not satisfied).

(b) Since $-1\le \sin x\sin y\le 1$, it follows that $-1\le f_2(x,y)\le 1$. The absolute maximum of f_2 is 1, and occurs at the points in S where both $\sin x=1$ and $\sin y=1$ or both $\sin x=-1$ and $\sin y=-1$: $(x=\pi/2, y=\pi/2)$ and $(x=-\pi/2, y=-\pi/2)$. The absolute minimum of f_2 is -1, and occurs at the points $(x=-\pi/2, y=\pi/2)$ and $(x=\pi/2, y=-\pi/2)$ in S.

(c) If we remove the assumption on closeness of S, the statement of the Theorem 21 might, or might not hold.

39. The domain of f is $\mathbb{R}^2$. If $x, y > 0$, then $f(x,y) = |x| + |y| = x + y$. The system $f_x = 1$, $f_y = 1$ gives no critical points. If $x > 0$ and $y < 0$ then $f(x,y) = |x| + |y| = x - y$. Again, $f_x = 1$, $f_y = -1$ gives no critical points. In the same way we check the remaining two cases, not getting any critical points. Note that f_x does not exist when $x = 0$, and f_y does not exist when $y = 0$. Thus, all points of the form $(x, 0)$ and $(0, y)$ where x and y are real numbers, are critical points of f.

41. We are looking for the line $y = mx + b$ which best fits the given data, in the sense that m and b minimize the function

$$\begin{aligned} f(m,b) &= ((-2m+b)-2)^2 + ((1m+b)-1)^2 + ((0m+b)-4)^2 \\ &= (-2m+b-2)^2 + (m+b-1)^2 + (b-4)^2 \end{aligned}$$

The partial derivatives of f are

$$\begin{aligned} f_m &= 2(-2m+b-2)(-2) + 2(m+b-1) = 10m - 2b + 6 \\ f_b &= 2(-2m+b-2) + 2(m+b-1) + 2(b-4) = -2m + 6b - 14 \end{aligned}$$

To find critical points, we solve the system (divide both $f_m = 0$ and $f_b = 0$ by 2):

$$\begin{aligned} 5m - b + 3 &= 0 \\ -m + 3b - 7 &= 0 \end{aligned}$$

Multiplying the first equation by 3 and adding to the second, we get $14m + 2 = 0$ and $m = -1/7$. Substituting $m = -1/7$ into the first equation, we get $-5/7 - b + 3 = 0$, i.e., $b = 16/7$. Thus, there is only one critical point, $(m = -1/7, b = 16/7)$. The regression line is $y = -x/7 + 16/7$.

To show that this critical point yields a minimum for f, we apply the second derivatives test. From $f_{mm} = 10$, $f_{mb} = f_{bm} = -2$, and $f_{bb} = 6$, we compute $D = f_{mm}f_{bb} - f_{mb}^2 = (10)(6) - (-2)^2 = 56 > 0$ Since $f_{mm} = 10 > 0$, the critical point we obtained is indeed a minimum.

43. We are looking for the line $y = mx + b$ which best fits the given data, in the sense that m and b minimize the function

$$\begin{aligned} f(m,b) &= ((0m+b)-0)^2 + ((0m+b)-1)^2 + ((1m+b)-1)^2 + ((2m+b)-2)^2 \\ &= b^2 + (b-1)^2 + (m+b-1)^2 + (2m+b-2)^2 \end{aligned}$$

The partial derivatives of f are

$$\begin{aligned} f_m &= 2(m+b-1) + 2(2m+b-2)(2) = 10m + 6b - 10 \\ f_b &= 2b + 2(b-1) + 2(m+b-1) + 2(2m+b-2) = 6m + 8b - 8 \end{aligned}$$

To find critical points, we solve the system (divide both $f_m = 0$ and $f_b = 0$ by 2):

$$\begin{aligned} 5m + 3b - 5 &= 0 \\ 3m + 4b - 4 &= 0 \end{aligned}$$

Multyplying the first equation by -4, the second equation by 3 and adding them up we get $-11m + 8 = 0$ and $m = 8/11$. Substituting $m = 8/11$ into the second equation, we get $24/11 + 4b - 4 = 0$, i.e., $4b = 20/11$ and $b = 5/11$. Thus, there is only one critical point, $(m = 8/11, b = 5/11)$. The regression line is $y = 8x/11 + 5/11$.

To show that this critical point yields a minimum for f, we apply the second derivatives test. From $f_{mm} = 10$, $f_{mb} = f_{bm} = 6$, and $f_{bb} = 8$, we compute $D = f_{mm}f_{bb} - f_{mb}^2 = (10)(8) - (6)^2 = 44 > 0$ Since $f_{mm} = 10 > 0$, the critical point we obtained is indeed a minimum.

45. Define the function

$$f(m,b) = ((mx_1 + b) - y_1)^2 + ((mx_2 + b) - y_2)^2 + \cdots + ((mx_n + b) - y_n)^2$$

To find critical points, we solve the following equations:

$$\begin{aligned}
f_m &= 2((mx_1+b)-y_1)x_1 + 2((mx_2+b)-y_2)x_2 + \cdots + 2((mx_n+b)-y_n)x_n \\
&= 2\left(mx_1^2 + bx_1 - x_1y_1 + mx_2^2 + bx_2 - x_2y_2 + \cdots + mx_n^2 + bx_n - x_ny_n\right) \\
&= 2\left(m(x_1^2+x_2^2+\cdots+x_n^2) + b(x_1+x_2+\cdots+x_n) - (x_1y_1+x_2y_2+\cdots+x_ny_n)\right) = 0
\end{aligned}$$

and

$$\begin{aligned}
f_b &= 2((mx_1+b)-y_1) + 2((mx_2+b)-y_2) + \cdots + 2((mx_n+b)-y_n) \\
&= 2m(x_1+x_2+\cdots+x_n) + 2(b+b+\cdots+b) - 2(y_1+y_2+\cdots+y_n) = 0
\end{aligned}$$

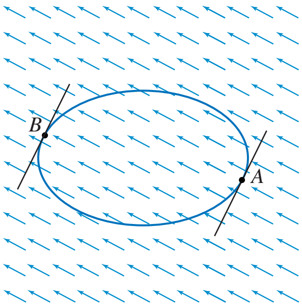
(in the second term b is added n times). Dividing both equations by 2 we obtain

$$\begin{aligned}
m(x_1^2+x_2^2+\cdots+x_n^2) + b(x_1+x_2+\cdots+x_n) &= x_1y_1+x_2y_2+\cdots+x_ny_n \\
m(x_1+x_2+\cdots+x_n) + bn &= y_1+y_2+\cdots+y_n
\end{aligned}$$

Section 11 Optimization with Constraints

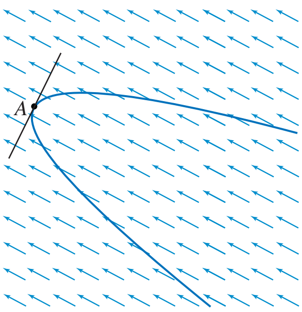
1. The level curves of f are concentric circles $(x^2+y^2)^{-1}=c$, i.e., $x^2+y^2=1/c$, where $c>0$. The constraint line $y=2x$ intersects all level curves orthogonally (i.e., at the point where $y=2x$ intersects a level curve, the line $y=2x$ is perpendicular to the tangent to the constraint curve). Consequently, the constraint curve is never parallel to the tangent to a level curve, and so f does not have a minimum or maximum subject to the constraint $2x-y=0$.

3. We are looking for the points on the constraint curve where the constraint curve (actually the tangent to the constraint curve) is perpendicular to the gradient of f. There are two such points, their approximate location is indicated in the figure below.



Recall that the gradient vectors indicate the direction of the (largest) increase in the values of f. At the points on the constraint curve near A, the values of f are larger than $f(A)$, since the curve moves in the direction of the gradient. Thus, f has a minimum at A subject to the given constraint. Near B, the constraint curve moves opposite of the gradient, and so the values of f along it decrease. Thus, $f(B)$ is a maximum of f subject to the given constraint.

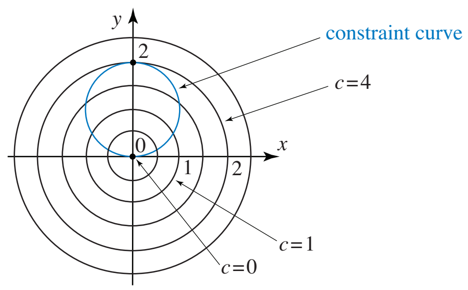
5. We are looking for the points on the constraint curve where the constraint curve (i.e., the tangent to the constraint curve) is perpendicular to the gradient of f. One point, labeled A in the figure below, satisfies this property.



Recall that the gradient vectors indicate the direction of the (largest) increase in the values of f. Walking away from A along the constraint curve, we move in the direction opposite of the gradient, and so the values of f decrease. Thus, $f(A)$ is a maximum of f subject to the given constraint.

7. If the two constraints are identical, then it's actually a single constraint. Otherwise (because they are lines), the two constraint curves either intersect at one point (in which case the set of values of f among which we need to pick the smallest and the largest values is reduced to one value), or do not intersect at all (in which case the set of values of f among which we need to pick the smallest and the largest values is empty).

9. The level curves of f are the circles $x^2+y^2=c$ (of radius $\sqrt{c}$, centred at the origin). Several level curves and the constraint curve are shown in the figure below.



(a) Walking around the constraint curve, we meet the level curves of values between 0 and 4. Thus the minimum of f subject to the given constraint is 0 (occurs at $(0,0)$), and the maximum is 4 (occurs at $(0,2)$).

(b) Let $g(x,y) = x^2 + (y-1)^2$; the constraint can be written as $g(x,y) = 1$. Note that both f and g are polynomials, and thus all of their partial derivatives are continuous. We compute $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2(y-1)\mathbf{j}$. The equation $\nabla f = \lambda \nabla g$ implies that $2x = 2x\lambda$ and $2y = 2(y-1)\lambda$.

Rewriting the first equation as $2x(\lambda - 1) = 0$, the solutions are $x = 0$ (in which case λ can be any real number) or $\lambda = 1$. Substituting $\lambda = 1$ into $2y = 2(y-1)\lambda$, we get $2y = 2y - 2$ and $0 = -2$. Thus, λ cannot be 1; so, $x = 0$.

Substituting $x = 0$ into the constraint equation, we get $(y-1)^2 = 1$, $y^2 - 2y + 1 = 1$ and $y^2 - 2y = 0$. Thus, $y = 0$ and $y = 2$, and we obtained two candidates for extremes, $(0,0)$ and $(0,2)$.

From $\nabla g = 2x\mathbf{i} + 2(y-1)\mathbf{j} = \mathbf{0}$ it follows that $x = 0$ and $y = 1$; i.e., ∇g is zero only at $(0,1)$. However, $(0,1)$ does not belong to the constraint curve (checking part (b) of Algorithm 2). The constraint curve is a circle (which has no endpoints), so we do not get any new candidates for extreme values from part (c) of Algorithm 2. Finally, the circle is a closed and bounded set, so assumption (2) is satisfied.

We compute $f(0,0) = 0$ and $f(2,0) = 4$ and conclude that $f(2,0) = 4$ is the maximum and $f(0,0) = 0$ is the minimum of f subject to the given constraint.

11. Let $g(x,y) = x^2 + y^2$; the constraint can be written as $g(x,y) = 9$. Both f and g are polynomials, and thus all of their partial derivatives are continuous. We find $\nabla f = 3x^2\mathbf{i} + 3y^2\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$. The equation $\nabla f = \lambda \nabla g$ implies that

$$3x^2 = 2x\lambda \quad \text{and} \quad x(3x - 2\lambda) = 0$$
$$3y^2 = 2y\lambda \quad \text{and} \quad y(3y - 2\lambda) = 0$$

The first equation implies that either $x = 0$ or $3x = 2\lambda$; the second equation implies that either $y = 0$ or $3y = 2\lambda$.

Substituting $x = 0$ into the constraint equation we get $y^2 = 9$ and $y = \pm 3$. Likewise, $y = 0$ implies that $x = \pm 3$. Thus, $(0,3)$, $(0,-3)$, $(3,0)$ and $(-3,0)$ are candidates for extreme values.

Substituting $x = 2\lambda/3$ and $y = 2\lambda/3$ into the constraint, we obtain

$$\frac{4\lambda^2}{9} + \frac{4\lambda^2}{9} = 9$$

i.e., $8\lambda^2 = 81$ and $\lambda = \pm 9/\sqrt{8}$. In this case,

$$x = y = \frac{2}{3}\lambda = \pm\frac{2}{3}\frac{9}{\sqrt{8}} = \pm\frac{2}{3}\frac{9}{2\sqrt{2}} = \pm\frac{3}{\sqrt{2}}$$

Thus, there are two more candidates for the extremes: $(3/\sqrt{2}, 3/\sqrt{2})$ and $(-3/\sqrt{2}, -3/\sqrt{2})$.

From $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} = \mathbf{0}$ it follows that $x = 0$ and $y = 0$; i.e., ∇g is zero at the origin only. However, $(0,0)$ does not belong to the constraint curve (we are checking part (b) of Algorithm 2). The constraint curve is a circle (which has no endpoints), so we do not get any new candidates for extreme values from part (c) of Algorithm 2. Finally, the circle is a closed and bounded set, so assumption (2) is satisfied.

The values of f at the points we obtained are: $f(0,3) = 3^3 = 27$, $f(0,-3) = (-3)^3 = -27$, $f(3,0) = 3^3 = 27$, $f(-3,0) = (-3)^3 = -27$,

$$f(3/\sqrt{2}, 3/\sqrt{2}) = \frac{27}{2\sqrt{2}} + \frac{27}{2\sqrt{2}} = \frac{54}{2\sqrt{2}} = \frac{27}{\sqrt{2}}$$

and, similarly, $f(-3/\sqrt{2}, -3/\sqrt{2}) = -27/\sqrt{2}$. Thus $f(0,3) = 27$, and $f(3,0) = 27$, are the maximum, and $f(0,-3) = -27$, and $f(-3,0) = -27$, are the minimum of f subject to the given constraint.

13. Let $g(x,y) = 2x - y$; the constraint can be written as $g(x,y) = 0$. Both f and g are polynomials, and thus all of their partial derivatives are continuous. We find $\nabla f = 2x\mathbf{i} - 2y\mathbf{j}$ and $\nabla g = 2\mathbf{i} - \mathbf{j}$. The equation $\nabla f = \lambda \nabla g$ implies that

$$2x = 2\lambda \quad \text{and} \quad x = \lambda$$
$$-2y = -\lambda \quad \text{and} \quad 2y = \lambda$$

Combining the two equations, we get $x = 2y$. Substituting $x = 2y$ into the constraint, we get $2(2y) - y = 0$, and $y = 0$. Thus, $x = 0$ as well, and so $(0,0)$ is a candidate for an extreme value.

Note that $\nabla g = 2\mathbf{i} - \mathbf{j} \neq \mathbf{0}$; as well, the constraint curve (the line $y = 2x$) has no endpoints (checking parts (b) and (c) of Algorithm 2). Thus, we do not get any new candidates for extreme values. The line is not a bounded set, so assumption (2) is not satisfied (which means that f might, or might not have extreme values subject to the given constraint).

The value of f at the only point we found is $f(0,0) = 0$. On the constraint curve $y = 2x$, f is equal to $f(x, y = 2x) = x^2 - (2x)^2 = -3x^2$. Thus, $f(0,0)$ is a maximum. The expression $-3x^2$ can be made smaller than any negative number, thus f does not have a minimum subject to the given constraint.

15. Let $g(x,y) = x^2 - y^2$; the constraint can be written as $g(x,y) = 1$. Its graph is a hyperbola with asymptotes $y = \pm x$ and x-intercepts $(\pm 1, 0)$. Both f and g are polynomials, and thus all of their partial derivatives are continuous. We find $\nabla f = 2xy^2\mathbf{i} + 2x^2y\mathbf{j}$ and $\nabla g = 2x\mathbf{i} - 2y\mathbf{j}$. The equation $\nabla f = \lambda \nabla g$ implies that

$$2xy^2 = 2x\lambda \quad \text{and} \quad x(y^2 - \lambda) = 0$$
$$2x^2y = -2y\lambda \quad \text{and} \quad y(x^2 + \lambda) = 0$$

The first equation implies that either $x = 0$ or $y^2 = \lambda$; the second equation implies that either $y = 0$ or $x^2 = -\lambda$.

Substituting $x = 0$ into the constraint equation we get $-y^2 = 1$, which has no solutions in real numbers. When $y = 0$, then $x^2 = 1$ and $x = \pm 1$. Thus, $(1,0)$ and $(-1,0)$ are candidates for extreme values.

Substituting $x^2 = -\lambda$ and $y^2 = \lambda$ into the constraint equation $x^2 - y^2 = 1$, we get $-\lambda - (\lambda) = 1$, i.e., $\lambda = -1/2$. From $x^2 = -\lambda$ we conclude that there are no solutions for x.

From $\nabla g = 2x\mathbf{i} - 2y\mathbf{j} = \mathbf{0}$ it follows that $x = 0$ and $y = 0$; i.e., ∇g is zero at the origin only. However, $(0,0)$ does not belong to the constraint curve (we are checking part (b) of Algorithm 2). The constraint curve has no endpoints, so we do not get any new candidates for extreme values from part (c) of Algorithm 2. The hyperbola $x^2 - y^2 = 1$ is not a bounded set, so assumption (2) is not satisfied (which means that f might, or might not have extreme values subject to the given constraint).

The values of f at the points we obtained are: $f(1,0) = 0$ and $f(-1,0) = 0$. Since $f(x,y) = x^2y^2 \geq 0$, we conclude that $f(1,0) = 0$ and $f(-1,0) = 0$ represent the minimum of f subject to the given constraint. The function f has no maximum, as its values can be made larger than any number. Alternatively, substituting $y^2 = x^2 - 1$ (the constraint) into $f(x,y) = x^2y^2$, we get $f(x) = x^2(x^2 - 1)$, which approaches infinity as x approaches infinity.

17. Let $g(x,y) = x^2 + 2y^2$; the constraint can be written as $g(x,y) = 2$. Writing the constraint as $x^2/2 + y^2 = 1$, we see that its graph is an ellipse. Both f and g are polynomials, and thus all of their partial derivatives are continuous. We find $\nabla f = y\mathbf{i} + (x+1)\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 4y\mathbf{j}$. The equation $\nabla f = \lambda \nabla g$ implies that $y = 2x\lambda$ and $x + 1 = 4y\lambda$. Computing λ from both equations, we get $\lambda = y/2x$ and $\lambda = (x+1)/4y$; thus

$$\frac{y}{2x} = \frac{x+1}{4y}$$

and $4y^2 = 2x(x+1)$ or $2y^2 = x^2 + x$. Substituting $2y^2 = x^2 + x$ into the constraint equation we get $x^2 + (x^2 + x) = 2$, and $2x^2 + x - 2 = 0$. Using the quadratic formula,

$$x = \frac{-1 \pm \sqrt{17}}{4}$$

and so $x \approx 0.78078$ and $x \approx -1.28078$. Now we use $2y^2 = x^2 + x$ to find y. When $x \approx 0.78078$ then $2y^2 \approx 1.39040$ and $y \approx \pm 0.83379$. When $x \approx -1.28078$ then $2y^2 \approx 0.35962$ and $y \approx \pm 0.42404$. Thus, $(0.78078, 0.83379)$, $(0.78078, -0.83379)$, $(-1.28078, 0.42404)$ and $(-1.28078, -0.42404)$ are candidates for the extreme values.

From $\nabla g = 2x\mathbf{i} + 4y\mathbf{j} = \mathbf{0}$ it follows that $x = 0$ and $y = 0$; i.e., ∇g is zero at the origin only. However, $(0, 0)$ does not belong to the constraint curve (we are checking part (b) of Algorithm 2). The constraint curve is an ellipse (which has no endpoints), so we do not get any new candidates for extreme values from part (c) of Algorithm 2. Finally, the circle is a closed and bounded set, so assumption (2) is satisfied.

The values of f at the points we obtained are:

$$\begin{aligned} f(0.78078, 0.83379) &\approx 1.48480 \\ f(0.78078, -0.83379) &\approx -1.48480 \\ f(-1.28078, 0.42404) &\approx -0.11906 \\ f(-1.28078, 0.42404) &\approx 0.11906 \end{aligned}$$

Thus, $f(0.78078, 0.83379) \approx 1.48480$ is the maximum, and $f(0.78078, -0.83379) \approx -1.48480$ the minimum of f subject to the given constraint.

19. We are asked to find a minimum of $T(x, y) = 2x^2 + y^2 + 120$ subject to $x^2 + y^2 = 6$.

Let $g(x, y) = x^2 + y^2$; the constraint can be written as $g(x, y) = 6$. Both T and g are polynomials, and thus all of their partial derivatives are continuous. We find $\nabla T = 4x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$. The equation $\nabla T = \lambda \nabla g$ implies that

$$\begin{aligned} 4x &= 2x\lambda \quad \text{and} \quad x(2-\lambda) = 0 \\ 2y &= 2y\lambda \quad \text{and} \quad y(1-\lambda) = 0 \end{aligned}$$

The two equations imply that $x = 0$ and $y = 0$.

Substituting $x = 0$ into the constraint equation we get $y^2 = 6$ and $y = \pm\sqrt{6}$. Likewise, $y = 0$ implies that $x = \pm\sqrt{6}$. Thus, $(0, \sqrt{6})$, $(0, -\sqrt{6})$, $(\sqrt{6}, 0)$ and $(-\sqrt{6}, 0)$ are candidates for the extreme values.

From $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} = \mathbf{0}$ it follows that $x = 0$ and $y = 0$; i.e., ∇g is zero at the origin only. However, $(0, 0)$ does not belong to the constraint curve (we are checking part (b) of Algorithm 2). The constraint curve is a circle (which has no endpoints), so we do not get any new candidates for extreme values from part (c) of Algorithm 2. Finally, the circle is a closed and bounded set, so assumption (2) is satisfied.

The values of the temperature T at the points we obtained are: $T(0, \sqrt{6}) = 126$, $T(0, -\sqrt{6}) = 126$, $T(\sqrt{6}, 0) = 132$, and $T(-\sqrt{6}, 0) = 132$. Thus, the coldest points on the rim are $(0, \sqrt{6})$ and $(0, -\sqrt{6})$, where the temperature is $T = 126$.