Contributions to Stable Model Theory

By

Ehud Hrushovski

A.B. (University of California) 1982

DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

in the

GRADUATE DIVISION

OF THE

UNIVERSITY OF CALIFORNIA, BERKELEY

Approved: [illegible] 4/11/86

Chairman .... Date

[illegible] 4/16/86

[illegible] 4-22-86

DOCTORAL DEGREE CONFERRED

MAY 16, 1986

ACKNOWLEDGEMENTS

I feel very lucky to have been a student of Leo Harrington's. His depth and breadth of vision are on a scale that I did not know existed. But he is able to know what is major without making anything seem minor, and to create a sense of excitement at every step of the way. Learning from him has been a great pleasure.

I was supported financially by a Berkeley Graduate Fellowship in 82/83 and a National Science Foundation Graduate Fellowship in 83/84-85/86.

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Chapter 1: Introduction

The thesis is organized around the solution of several problems in stable model theory. In chapter 2 we prove Kueker's conjecture for stable theories: a countable stable theory whose every uncountable model is $\aleph_0$-saturated must be either $\aleph_0$-categorical or $\aleph_1$-categorical. The conjecture is still open for unstable theories, but in §2.3 we prove it for theories interpreting linear orderings; this is our only excursion outside the domain of stability. Chapter 3 contains a solution of Shelah's problem on unidimensional theories, proving for example that a countable theory with at most one $\aleph_1$-saturated model in each power must be superstable. In chapter 4 we find an absolute finite bound (3) on the number of parameters needed to witness non-orthogonality between a regular type and a normal set. Chapter 5 contains an analysis of locally modular regular types, solving all purely geometric problems about them. Chapter 6 is somewhat disjoint from the rest of the thesis. There the techniques of §1 are used to show that the main problems about countable models of stable theories are easy for finitely based theories.

The results are presented as consequences of a more general technology, which can be viewed as a beginning of a theory of "almost-orthogonality". It is presented in §§2.1, 3.1-3.3, theorem 3.4.2, and §4. (Much of §5 addresses the same problem for locally modular regular types, but it is not known at present what part this will play in a general theory.) The starting point is Shelah's theory of regular and especially semi-regular types; §2.1 and §3.2 are largely presentations of his theory. The unexpected (but central) new feature is the entry of the theory of stable groups into the abstract context. The theorem on unidimensional theories, the finite bound result of chapter 4, and theorem 5.3 have no known proof that does not use stable groups. (in the case of theorem 4.1, in view of its full statement, it is difficult to imagine a possible one.) To prepare for this we prove some facts about stable groups in general, generalizing a definablity result of Poizat's in §3.1, and the Lascar-Berline theory in §3.3. The interpretation techniques themselves can be found in theorems 3.4.1 and 4.1(version 2), and in theorem 5.2 and proposition 5.2.10.

Each chapter has a more detailed introduction.

CONVENTIONS AND NOTATION

The rest of this chapter is intended to fix notation and conventions. It includes, however, an exposition of basic stability theory, as presented in Harrington's model theory course in Berkeley in 1982/83. The approach is so direct as to make it simply inefficient to state the definitions without proving the theorems as well. The results are of course Shelah's. We will adopt the notation of [M] almost entirely, with one or two exceptions listed below. We also include a summary of the basic facts on generic types in stable groups, due to Poizat.

First Order Theories.

L is a possibly many-sorted language. T will always be a complete theory in L. As we will never be concerned with the process of generating formulas, we will consider every formula of L to be atomic. (Thus submodels are always elementary.) As in [M], one adds "imaginary" sorts to L to obtain a language L^{eq} and a complete theory T^{eq} with the following property: for every 0-definable set S of n-tuples, and every 0-definable relation E such that $T \vdash$"E is an equivalence relation on S", there exists a 0-definable function f with domain S such that E is the kernel of f. This property is called "$T=T^{eq}$". It is easy to verify that it remains true for expansions by constants. Every model M of T extends canonically to a model M^{eq} of T^{eq}. We will work in T^{eq} throughout, noting it explicitly only at the level of theorems. As a derived convention, we will not in general distinguish notationally between elements and n-tuples.

We fix from the start a universal domain $\mathbb{C}$. Let μ be a cardinal greater than any which concern us at a given moment. $\mathbb{C}$ is chosen to be μ-saturated, μ-homogeneous, and as close to saturated as is neccessary. (T will mostly be stable, so $\mathbb{C}$ can be chosen saturated.) All models and substructures are assumed to be substructures of $\mathbb{C}$, of cardinality $<\mu$. We will constantly use "saturated Galois theory." $\mathrm{Aut}(\mathbb{C})$ is the group of automorphisms of $\mathbb{C}$. By the conventions on atomic formulas and on cardinalities, every isomorphism between substructures of $\mathbb{C}$ extends to some $\sigma \in \mathrm{Aut}(\mathbb{C})$. Build a model of set theory starting with $\mathbb{C}$ as a set of atoms and the relations on $\mathbb{C}$ distinguished classes, and work in it. Then $\mathrm{Aut}(\mathbb{C})$ acts on everything. Define: $\mathrm{Aut}(\mathbb{C}/A)=\{\sigma \in \mathrm{Aut}(\mathbb{C}): \sigma a=a \text{ for } a \in A\}$

$\mathrm{dcl}^*(A)=\{x: \sigma x=x \text{ for all } \sigma \in \mathrm{Aut}(\mathbb{C}/A)\}$

$\mathrm{acl}^*(A)=\{x:$ the orbit of a under $\mathrm{Aut}(\mathbb{C}/A)$ is finite$\}$.

$\mathrm{dcl}(A)=\mathrm{dcl}^*(A)\cap\mathbb{C}^{eq}$. $\mathrm{acl}(A)=\mathrm{acl}^*(A)\cap\mathbb{C}^{eq}$.

A set $D\subset\mathbb{C}$ is called A-definable if it has the form $\{x \in \mathbb{C}: \mathbb{C} \models \varphi(x,a_1,\dots,a_n)\}$ for some $\varphi \in L$ and $a_1,\dots,a_n \in A$. D is definable if it is A-definable for some A. It is easy to see that D is A-definable iff D is definable and $D \in \mathrm{dcl}^*(A)$. 0-definable means $\emptyset$-definable. A $\boldsymbol{\Lambda}$-definable set is an intersection of definable sets.

$L_x(\mathbb{C})$ is the Tarski-Lindenbaum algebra of all A-definable formulas $\varphi(x,a)$ up to logical equivalence. Equivalently, it is the algebra of A-definable subsets of $\mathbb{C}$, the sort of whose elements is the sort of x.

<u>STABILITY</u>

A family of subsets of $\mathbb{C}$ will be said to be uniformly definable if it has the form $\{D_a: a \in F\}$, where F and $\{(a,b): b \in D_a\}$ are both 0-definable. A subalgebra of $L_x(\mathbb{C})$ is called uniform if it is generated (as a Boolean algebra) by a finite set of uniform families. The letter Δ will be reserved for such families. A Δ-type is an ultrafilter on Δ. One defines a notion of rank on partial types P as follows:

Definition $rk_\Delta(X)=-1$ iff P is inconsistent.

For an ordinal α, $rk_\Delta(X)=\alpha$ if the number of Δ-types consistent with $P \cup \{\sim D: D \in \Delta$ and $rk(\{D\})=\beta$ for some $\beta<\alpha\}$ is finite and non-zero. This number is called the Δ-multiplicity of X, or $Mult_\Delta(X)$.

$rk_\Delta(D)=rk_\Delta(\{D\})$ for a definable set D.

Definition T is stable if $rk_\Delta(F)$ exists for every F and Δ.

An easy argument shows that this is equivalent to the definition in terms of the number of types (which we will not use.)

From now on assume that T is stable. Fix a uniform Δ. Call $\delta(x,y)$ a D-formula if $\varphi(x,b) \in \Delta$ for all b. Δ generated by a finite set of Δ-formulas $\delta_i(x,y)$. Let r-M denote the pair: Δ-rank, Δ-multiplicity, and order the set of pairs of ordinals lexicographically. The following observation is the basis of everything. (In particular,it follows easily from (b) that Δ-rank is always finite.)

Observation 1

(a) Given $\Delta \subset L_x(\mathbb{C})$, a formula $\varphi(x,y)$, an ordinal α, and an integer m,

$\{b: r\text{-}M(\varphi(x,b)) \geq \alpha,m\}$ is a $\mathbf{\Lambda}$-definable set.

(b) Suppose $r\text{-}M(\rho(x))=\alpha,m$, and $\psi(x,b) \in \Delta$ for all b. Then

$\{b: r\text{-}M(\rho(x) \& \psi(x,b)) \geq \alpha,m\}$ is a definable set.

Proof

(a) The assertion follows by induction on α and m from the relations:

i) $r\text{-}M(\varphi(x,b)) \geq 0,1$ <u>iff</u> $\exists x \varphi(x,b)$

ii) $r\text{-}M(\varphi(x,b)) \geq \alpha,1$ <u>iff</u> for all $\beta < \alpha$ and all m, $r\text{-}M(\varphi(x,b) \geq \beta,m$

iii) For $\alpha > 0$ and $m > 1$, $r\text{-}M(\varphi(x,b)) \geq \alpha,m$ <u>iff</u> for some δ_i and some $k,l \geq 1$ with $k+l=m$,

$(\exists b')(r\text{-}M(\varphi(x,b) \& \delta_i(x,b')) \geq \alpha,k \ \& \ r\text{-}M(\varphi(x,b) \& \sim \delta_i(x,b')) \geq \alpha,l))$.

(b) Let Z be the set in question. The complement $\sim Z$ of Z (inside the relevant sort) is $\{b: r\text{-}M(\rho(x) \& \sim \psi(x,b \geq \alpha,1\}$. By (a), Z and $\sim Z$ are both $\mathbf{\Lambda}$-definable. Hence both are definable.

Corollary and Definition 2 (Definability Theorem) Let p be a (complete) Δ-type, and let $\delta(x,y)$ be a Δ-formula. Then there exists a formula $\psi(y)$ such that $\delta(x,b) \in p$ iff $\models \psi(b)$. ψ is called the p-definition of δ, and is denoted by $(d_p x)\delta(x,y)$. (It is obviously well defined up to logical equivalence.) $(d_p x)$ can be thought of as a quantifier, and pronounced as: "for generic x realizing p".

proof Choose $\psi(x) \in p$ with $r\text{-}M(\varphi)=\alpha,m$ as small as possible. If $m>1$ then there exists $p' \neq p$ with $\psi \in p'$ and $rk_\Delta(p')=\alpha$, and by choosing $\psi' \in p-p'$ it is easy to lower m; so m=1. Now $\{b: \delta(x,b)\in p\}=\{b: r\text{-}M(\psi(x)\ \&\ \delta(x,b))\geq\alpha,1\}$ is definable by (b).

Remark and Definition 3 Let p be a Δ-type. Say Δ is generated by $\{\delta_i(x,\bar{b}): 1\leq i\leq n, \bar{b}\in\mathbb{C}\}$. Then clearly p and $((d_p x)\delta_i(x,\bar{y}) : 1\leq i\leq n)$ are equi-definable. Hence there exists an element e of $\mathbb{C}^{eq}$ such that $dcl^*(p)=dcl^*(e)$. dcl(e) is called the canonical base of p, or Cb(p). (Being equal to dcl(p), it does not depend on the choices made.)

Corollary 4 (Existence of a definable extension) Let q be a complete type over A and assume A=acl(A). Then there exists a Δ-type r consistent with q such that $r\in dcl^*(A)$. (Moreover, $rk_\Delta(r)=rk_\Delta(q)$.)

Proof Say $r\text{-}M(q)=\alpha,m$. Let $r_1,\ldots,r_m$ be the m Δ-types of rank α consistent with q. Then $r=r_1$ has at most m conjugates over A. (I.e. the orbit of r under $Aut(\mathbb{C}/A)$ has cardinality at most m.) Let e be such that dcl(e)=Cb(r). Then e has at most m conjugates over A. Since A=acl(A), $e\in A$. Thus r is A-definable.

Lemma 5 (Symmetry) Let $\Delta\subset L_x(\mathbb{C})$, $\nabla\subset L_y(\mathbb{C})$ be two uniform subalgebras. Suppose $\delta(x,y)$ is a Δ-formula (in x) and a ∇-formula (in y). Let p be an A-definable Δ-type, consistent with $p_0=tp(a/A)$, and let q be an A-definable ∇-type consistent with $q_0=tp(b/A)$. Then $\models(d_q y)(\varphi(a,y))$ iff $\models(d_p x)\varphi(x,b)$.

Proof Suppose otherwise. Then (say) $\models(d_q y)(\varphi(a,y))$ and $\models(d_p x)\sim\varphi(x,b)$. Find $a_1,\ldots b_1,\ldots$ inductively so that a_n realizes $p_0\cup\{-\varphi(x,b_i): i\leq n\}$, b_n realizes $q\cup\{-\varphi(a_i,x): i<n\}$. This is possible since the (partial) types in

question are subsets of $p \cup p_0$ and $q \cup q_0$, which are consistent. But in the end $\varphi(a_i,b_j)$ holds iff $i<j$; this can easily be seen to contradict stability.

Corollary 6 (Uniqueness of the definable extension.) Let q be a complete type over $A=\mathrm{acl}(A)$. Then there exists a unique Δ-type r consistent with q and definable over A.

Proof Suppose r_1,r_2 are two. Let $\delta(x,y)$ be a Δ-formula, and let b be an element. Let $\nabla \subset L_y(\mathbb{C})$ be a uniform algebra such that $\delta(x,y)$ is a ∇-formula (in y). Let $q_0=\mathrm{tp}(b/A)$ and, using corollary 4, let q be a ∇-type in $\mathrm{dcl}^*(A)$ consistent with q_0. Let $a \models q_0$. Applying symmetry twice, we see that $\delta(x,b) \in r_1$ iff $\models (d_q y)\delta(a,y)$ iff $\delta(x,b) \in r_2$. Thus $r_1=r_2$.

Conclusion and definition 7 Let q be a complete type over $A=\mathrm{acl}(A)$. Then there exists a unique extension $\hat{q}$ of q to a complete type over $\mathbb{C}$ such that $\hat{q}$ is A-definable. $\hat{q}$ is called the non-forking extension of q to $\mathbb{C}$. One has $\mathrm{rk}_\Delta(\hat{q})=\mathrm{rk}_\Delta(q)$ for all Δ.

Proof For each Δ, there exists an A-definable Δ-type consistent with q, which also has the same Δ-rank. By uniqueness and the fact that the Δ's form a directed set, all these Δ-types must cohere.

The following definition deviates a little from [M].

Definitions A strong type p is a function assigning to each $A \subset \mathbb{C}$ a complete type over A, denoted $p|A$, such that $p|B$ extends $p|A$ if $B \supset A$.
If p_0 is a complete type over an algebraically closed set A, the strong type associated with p_0 is defined to be the map $A \to p|A$ where p is the non-

forking extension of p_0 to $\mathfrak{C}$. Usually p and p_0 will be notationally identified. stp(a/A) will be the strong type associated with tp(a/acl(A)). For a strong type p, $Cb(p)=_{def}dcl(p)$. It is clear that $dcl(p)=dcl((p|\Delta: \Delta))$; so $Cb(p)=\cup\{Cb(p|\Delta): \Delta\}$, and p is the non-forking extension to $\mathfrak{C}$ of $p|Cb(p))$. p is said to be based on A if $Cb(p)\subset acl(A)$, stationary over A if $Cb(p)\subset A$. $(d_p x)\varphi(x,\overline{y})$ is the formula that holds of $\overline{b}$ iff $\varphi(x,\overline{b})\in p|B\cup\{\overline{b}\}$. (So $d_p:L_{xy}(B)\to L_y(B)$ is a Boolean projection.)

Notation There are many names for forking. $A\perp B|C$ means: for every finite sequence a from A, $a\models stp(a/C)|B\cup C$. By the symmetry lemma, $A\perp B|C$ iff $B\perp A|C$. The elementary properties of this symbol are listed in [M] or [HH], and will be used without mention. The symbol is read: A and B are freely joined (or independent) over C. The base set C is omitted (here and in other symbols) if $C=\varnothing$, or if its identity is clear for some other reason. A set I of subsets of $\mathfrak{C}$ is independent (over C) if $A\perp\cup(I-\{A\})|C$ for every $A\in I$.

If p,q are strong types in variables x,y then by composing d_p,d_q one obtains another Boolean projection, and the associated type is called $p\otimes q$. (So if p=stp(c/B) and q=stp(d/B) and $c\perp d|B$ then $p\otimes q$=stp(cd/B).) $p\otimes\ldots\otimes p$ (n times) is denoted by $p^{(n)}$.)

It is easy to deduce the following facts.

8. Let S be a set and a an element. Then $Cb(stp(a/S))\subset acl(S)$ and has cardinality at most $|T|$. Thus $a\perp S|S_0$ for some $S_0\subset S$ with $card(S_0)\leq|T|$. In particular, if I is an independent set then $a\perp\cup\{I-I_0\}$ for some $I_0\subset I$ with $card(I_0)\leq card(T)$.

9. Let $D(x,y)$ be a formula and let $b_1,b_2,\ldots$ be an infinite independent sequence of realizations of a strong type p. Then $(d_p x)D(x,y)$ is equivalent to a positive Boolean combination of the formulas $D(x,b_i)$. In particular, $Cb(p)\subset dcl\{b_1,b_2,\ldots\}$.

10. Let D be a $\bigwedge$-definable set, and let $R(x_1,\ldots,x_n,\overline{b})$ be a ($\bigwedge$-)definable relation such that $R(x_1,\ldots,x_n,\overline{y})\Rightarrow \bigwedge_i D(x_i)$. Then R is ($\bigwedge$-)definable with parameters from D.

<u>Convention</u> A function of the type of an element will often be written as a function of the element, e.g. Cb(b/X) in place Cb(stp(b/X)). In the presence of such notions (ones invariant under definitional closure) sequences of elements, sequences of sequences, etc. will be confounded with their fields.

<u>STABLE GROUPS</u>

For us a stable group will be a $\bigwedge$- definable group in a stable structure. The notion of a generic type comes from [Pz], generalizing ideas of Zil'ber. We briefly give a version that works for $\bigwedge$-definable as well as outright definable groups. The proofs also work word for word for *-definable groups, which will be defined later. We will also need an (equally trivial) generalization to group actions.

Let G be a $\bigwedge$-definable group, S a $\bigwedge$-definable set, and let there be given a definable action of G on S. Given Δ, let $\Delta^*=\{\varphi(\sigma\cdot x,\overline{u}): \varphi(x,\overline{u})\in\Delta,\sigma\in G\}$. Δ^* may not be uniform, but it is contained in the uniform algebra $\{\varphi(v\cdot x,\overline{u}): \varphi\in\Delta\}$; so every definable set has finite Δ^*-rank. The advantage in considering Δ^*'s is that rank and multiplicity are translation invariant. Thus for example we can prove:

Lemma 11 ([BS]) For any Δ, the set of subgroups of G of the form $G \cap D$ (where D is the extension of a Δ-formula) has the ascending and descending chain conditions.

Proof Suppose for example $G_0 \supset G_1 \supset G_2 \ldots$ is a strictly descending chain of Δ-definable subgroups. Then for each n, G_n contains two disjoint cosets of G_{n+1}, each of which has the same Δ^* rank and multiplicity as G_{n+1}; so $r\text{-}M_{\Delta^*}(G_n) > r\text{-}M_{\Delta^*}(G_{n+1})$. This is a contradiction.

Definition In particular, there exists a unique smallest Δ-definable subgroup of G of finite index, called the Δ-connected component. The connected component of G is the intersection of all the Δ-connected components.

Definition Let G be a $\mathbf{\Lambda}$-definable group, S a $\mathbf{\Lambda}$-definable set, and let there be given a definable action of G on S. Then a strong type p (of elements of S) is *generic* for the action of G if for all $\sigma \in G$, if $x \models p|\sigma$ then $\sigma \cdot x \perp \sigma$. Note that G acts on the set of generic strong types: ${}^{\sigma}p = q$ if for $x \models p|\sigma$, $\sigma \cdot x \models q|\sigma$. A strong type of G is said to be generic if it is so with respect to the action on G on itself by left translation.

Facts

12) There always exists a generic type.

13a) If G acts transitively on S, then it acts transitively on the set of generic types of S.

b) The connected component of G acts trivially on the set of generics.

c) If G acts transitively on S, then for every Δ, $\{p|\Delta : p$ a generic type of $S\}$ is finite.

Proof

12) Using observation 1 and compactness, it is not hard to find a strong type p of elements of S with the following property: there is no strong type q (of the same sort) such that $rk_{\Delta^*}(q) \geq rk_{\Delta^*}(p)$ for all Δ with some of the inequalities strict. Clearly p is based on $\varnothing$. Let $\sigma \in G$, and let $a \models p|\sigma$. Then for any Δ, $rk_{\Delta^*}(\sigma \cdot a/\varnothing) \geq rk_{\Delta^*}(\sigma \cdot a/\sigma) = rk_{\Delta^*}(a/\sigma) = rk_{\Delta^*}(p)$. By the choice of p, equality holds everywhere, i.e. $\sigma \cdot a \perp \sigma$. Thus p is generic.

13)(b) Let p be a generic type. Let $Fix(p,\Delta) =_{def} \{\sigma : {}^{\sigma}p|\Delta^* = p|\Delta^*\}$. Since it is automatically true for any σ that $r\text{-}M_{\Delta^*}({}^{\sigma}p) = r\text{-}M_{\Delta^*}(p)$, $Fix(p,\Delta) = \{\sigma : (d_p x)\varphi(\sigma x)\}$ where φ is a formula of least r-M in p. Thus $Fix(p,\Delta)$ is a definable subgroup. Let $F = \cap\{Fix(p,\Delta) : p \text{ generic}, \Delta \text{ uniform}\}$. Then G/F acts faithfully on the set of generic types, and so its cardinality is bounded by a function the cardinality of the language (every generic type is based on $\varnothing$.) By compactness, each $Fix(p,\Delta)$ must have finite index, and so contain the connected component. This shows that the connected component G° acts trivially on the generics.

(a) Let p,q be generic. Let $a \models p$, $b \models q$, and choose σ such that $\sigma a = b$. Pick $\tau \in G^\circ$ generic for the action of G on itself by right translation, and $\tau \perp \{\sigma, a, b\}$. Then $\tau\sigma a = \tau b \models {}^{\tau}q = q$. But $\tau\sigma \perp a|\sigma$, and $\tau\sigma \perp \sigma$, so $\tau\sigma \perp a$. Thus ${}^{\tau\sigma}p = q$. So G acts transitively on the generics. In particular for each Δ, all generics have the same Δ^*-rank.

(c) By the last sentence of the proof of (a), all generics have maximal Δ^*-rank for each Δ, i.e. $rk_{\Delta^*}(p)=rk_{\Delta^*}(S)$ for each Δ^*. By definition of rank, there are only finitely many possible Δ^*-types of this kind.

SUPERSTABILITY

The only fact from superstability theory that we will seriously use is that regular types are important. But we will work so close to a single regular type most of the time that the superstability assumption itself will not be needed. When we do use superstability we will be faithful to the notation of [M], the only addition being that $p \bowtie q$ denotes ($p \trianglelefteq q$ and $q \trianglelefteq p$), and similarly for $p \,{}^{a}\!\!\bowtie q$.

Chapter 2: Kueker's Conjecture for Stable Theories

If a countable theory T is categorical in some uncountable power, then every uncountable model of T is saturated in its own cardinality. (Morley) If T is $\aleph_0$-categorical, then every model of T is $\aleph_0$-saturated. (Ryll-Nardjewski.) Thus if T is categorical in some power, then every uncountable model of T is $\aleph_0$-saturated. Kueker's conjecture is that the converse is true. This has been proved by Lachlan for ω-stable theories, and by Buechler for superstable ones [Bu1]. Continuing this line, we prove it for stable theories (2.2). §2.1 contains a little general machinery. (It will be used again in §3.) In §2.3 the conjecture is proved for theories that interpret a linear ordering.

§2.1 A Decomposition Lemma

T is assumed stable, but not neccessarily countable. The point of the following theorem is that the hypothesis mentions only finitely based types. In (a) it could even have been restricted to 1-types p(x) that are based and stationary over a singleton.

Proposition 1

a) Assume every finitely based, non-algebraic type is non-orthogonal to some superstable definable set. Then T is superstable.

b) Let T be countable, and assume every finitely based, non-algebraic type is non-orthogonal to some totally transcendental definable set. Then T is ω-stable.

Here a definable set D is said to be superstable if $R^{\infty}(D)<\infty$, totally transcendental if D has ordinal Morley rank. The proposition will be obtained by analyzing the complexity of the notions of orthogonality between a type and a definable set, and the polar notion of a type analyzable in terms of a definable set. A strong type p is said to be foreign to a definable set D if for all sets B, all $a\models p|B$ and all $d\in D$, $a\perp d|B$.

Lemma 2 Let $D=D(y,b)$, $b\in M$, $p=\mathrm{stp}(c/M)$. Then (a)-(d) are equivalent:

(a) p is foreign to D

(b) for every formula $R(x,y)$ in $L(M)$ such that $R(x,y)\Rightarrow D(y)$, $R(c,y)$ is equivalent to a formula in $L(M)$.

(c) for every formula $R(x,y)$ in $L(M)$ such that $R(x,y)\Rightarrow D(y)$, if $R(c,y)$ has a solution then it has one in M.

(d) For every $R(x,uv,y)\in L$ such that $\models R(x,uv,y)\rightarrow D(y,v)$,
$\models (\forall u)(\ \ (d_px)(\exists y)R(x,ub,y) \equiv (\exists y)(d_px)R(x,ub,y)\ \ )$.

If T is countable, another equivalent formulation is

(e) There exists a model $N\supset M\cup\{c\}$ such that $D^M=D^N$.

Proof (b)$\Rightarrow$(c)$\Rightarrow$(d)$\Rightarrow$(a) involve no more than opening up the definitions, nor does (a)$\Rightarrow$(b) once one specifies that the requisite formula in $L(M)$ is $(d_px)R(x,y)$. If T is countable, then the omitting types theorem gives a precise syntactical equivalent to (e), and this turns out to be exactly (c).

We can draw one immediate conclusion in passing. Observe the form of (d): it describes a certain partial type, call it $t_{p\perp D}(v)$, such that $t_{p\perp D}(b)$ is true iff $p\perp D(y,b)$. This can be used to prove:

Corollary 3 If T has the finite cover property then in T^{eq} there exists an infinite definable set D such that every type based on $\varnothing$ is orthogonal to D.

Proof Recall the following definition of the finite cover property. Given a formula $D(x,y)$, let $i_D(y)=\{(\exists^{\geq n}x)D(x,y): n<\omega\}$. Then T does not have the fcp iff T is stable, and for all D, $i_D(b)$ is generated by a single formula. Suppose the conclusion fails; then $i_D(y) \cup \cup\{t_{p\perp D}(y) : p \text{ based on } \varnothing\}$ is inconsistent. By compactness, the same is true if $i_D(y)$ is replaced by some formula $(\exists^{\geq n}x)D(x,y)$. So if $\models(\exists^{\geq n}x)D(x,b)$ then $p\not\perp D(x,b)$ for some p based on $\varnothing$, whence $D(x,b)$ cannot be finite. This is the nfcp.

Now we come to the opposite notion. (The fact that it is that will be justified soon.) The discussion leading up to proposition 5 is a presentation of an idea from [Sh V.5-6].

Let **P** be a family of partial types, and write $a\models P|B$ if tp(a/B) extends some member of **P**. **P** is said to be based on E if it is invariant under the action of $\mathrm{Aut}(\mathfrak{C}/E)$. The main examples to have in mind: (1) **P** is a single partial type P (or a single formula D); (2) The class of conjugates of a given regular type p, with p strongly non-orthgonal to $\varnothing$. (I.e. no two conjugates of p are orthogonal. p is non-orthogonal to $\varnothing$ iff it is strongly non-orthgonal to acl($\varnothing$).) In these cases we will write P,D, or p for **P**. In case (2), Proposition 5 below is Shelah's existence theorem for semi-regular types.

Assume from now on that **P** is based on $\varnothing$.

Define:

- p is foreign to **P** if for all B, all $a\models p|B$, and all $c\models \mathbf{P}|B$, $a\perp c|B$.

• p is $\mathbf{P}$-internal if there exist B, $a \models p|B$, and elements $d_i \models \mathbf{P}|B$ such that $a \in dcl(B \cup \{d_i: i\})$.

• a is $\mathbf{P}$-analyzable (in λ steps) over A if there exists a sequence a_i $(i \leq \lambda)$ of elements of $dcl(a)$ such that, letting $A_i = A \cup \{a_j : j < i\}$, $stp(a_i/A_i)$ is $\mathbf{P}$-internal for each i, and $a_\lambda = a$.

Remark 4 If $\mathbf{P}$ consists of a single type q, then "p is foreign to $\mathbf{P}$" is much stronger (in general) than "$p \perp q$"; the first notion says that p is orthogonal not only to q, but to every forking extension of q as well. In general, p is foreign to $\mathbf{P}$ iff there exist $|T|^+$-saturated models M with $\dim_p M$ arbitrarily large compared to the cardinality of $\{a \in M: a \models \mathbf{P}\}$.

Notation $int_{\mathbf{P}}(A/B) = \{a \in dcl(A \cup B): stp(a/B)$ is $\mathbf{P}$-internal$\}$. $int_{\mathbf{P}}(A) = int_{\mathbf{P}}(A/\emptyset)$. $int_{\mathbf{P},\alpha}(A/B) = \{a \in dcl(A \cup B)$: a is $\mathbf{P}$-analyzable over B in $<\alpha$ steps$\}$. $int_{\mathbf{P},\infty}(A/B) = \cup_\alpha int_{\mathbf{P},\alpha}(A/B)$. $A_{\mathbf{P}} = int_{\mathbf{P},\infty}(A) = int_{\mathbf{P},\infty}(A/\emptyset)$.

Proposition 5 Assume $\mathbf{P}$ is based on $\emptyset$. Let $A \perp B$ and let E be a set such that $d \models \mathbf{P}|B$ for each $d \in E$. Then $A \perp B \cup E \mid int_{\mathbf{P}}(A)$.

Proof It is easy to reduce the proposition to the case where A is algebraically closed, and $E = \{d\}$ is a singleton. Assuming this, $Cb(Bd/A) \subset A$, and $A \perp B \cup \{d\} | X$ for any X such that $Cb(Bd/A) \subset X \subset A$. So the proposition says precisely that $Cb(Bd/A) \subset int_{\mathbf{P}}(A)$. In other words, letting $Y = Cb(Bd/A)$, we have to show that $stp(Y/A)$ is $\mathbf{P}$-internal. Let $B_1d_1, B_2d_2, \ldots$, be an infinite Morley sequence over A with $B_1d_1 = Bd$. By Fact 1.9, $Y \subset dcl(B_1 \cup B_2 \cup \ldots \cup \{d_1, d_2, \ldots\})$. Now $B_1, B_2, \ldots$ is an independent sequence over A, and each B_i is free from A over $\emptyset$. Thus $A \perp \cup_i B_i$. Since $Y \subset A$, $Y \perp \cup_i B_i$.

$d_i \models \mathbf{P} | B_i$ for each i because $\mathbf{P}$ is based on $\emptyset$. This shows that stp(Y/A) satisfies the definition of a $\mathbf{P}$-internal type.

This falls short of saying that $stp(A/int_{\mathbf{P}}(A))$ is foreign to $\mathbf{P}$. It does imply that $int_{\mathbf{P}}(a) \subset acl(\emptyset)$ iff $stp(a/\emptyset) \perp P$.

Corollary 6 (decomposition lemma.) $stp(A/A_{\mathbf{P}})$ is foreign to $\mathbf{P}$.

Remarks 7

(a) If a/A is $\mathbf{P}$-internal (or analyzable), and $A \subset B$, then a/B is $\mathbf{P}$-internal (analyzable.) This follows from the definitions by the standard arithmetic of forking. In particular, $stp(A_{\mathbf{P}}/X)$ is $\mathbf{P}$-analyzable for any X; so if stp(A/X) is foreign to $\mathbf{P}$ then $A_{\mathbf{P}} \subset acl(X)$. Thus:

(b) a is $\mathbf{P}$-analyzable over B iff for all $C \supset B$, if tp(a/C) is foreign to $\mathbf{P}$ then $a \in acl(C)$.

(c) It follows from (b) that the notion is parallelism invariant: if $a \perp B | B_0$ and a is $\mathbf{P}$-analyzable over B, then a is $\mathbf{P}$-analyzable over B_0. **(d)** For a moment, call a externally $\mathbf{P}$-analyzable over B if there exist a_i $(i \leq \lambda)$ such that $stp(a_i/B \cup \{a_j : j<i\})$ is $\mathbf{P}$-internal, and $a_\lambda = a$. (Without insisting that $a_i \in dcl(a)$.) Let $\mathbf{Q}$ be the set of all externally $\mathbf{P}$-anlyzable types. Then it is easy to prove (using induction on the length of the external analysis) that for all p, if p is foreign to $\mathbf{P}$ then p is foreign to $\mathbf{Q}$. Hence by (b), externally $\mathbf{P}$-analyzable = $\mathbf{P}$-analyzable. The notion of an external $\mathbf{P}$-analysis, and the various lemmas proved here (such as (b)), were really already present in [BL]. The new ingredient here is Shelah's lemma, showing that the analysis can be carried out entirely inside the definitional closure of the element being analyzed.

Proposition 8 Suppose **P** is a family of formulas. Then every **P**-analyzable type is **P**-analyzable in finitely many steps.
(More precise restatement: Suppose every partial type in **P** has the form $\{D(x,\bar{b})\}$ for some D. Suppose a is **P**-analyzable over B. Then for some $n<\omega$ a is **P**-analyzable over B in n steps.)

Proof The proposition is an easy combinatorial consequence of the following claim:
(*) if $stp(c/E)$ is **P**-internal then $stp(c/E_0)$ is **P**-internal for some finite $E_0 \subset E$.

Let $stp(c/E)$ be **P**-internal. Then there exists $B \supset E$, $c \perp B \mid E$ and elements $d_i \models \mathbf{P} \mid B$ $(i=1,\dots,n)$ such that $c \in dcl(B \cup \{d_i\}\text{'s})$. So there exists a definable function f and $b \in dcl(B)$ such that $c=f(b,d_1,\dots d_n)$. Since **P** is a family of formulas, there exist formulas $D_i(x,b_i) \in \mathbf{P}$ $(b_i \in dcl(B))$ such that $\models D_i(d_i,b_i)$ for each i. Let $q=stp(bb_1\dots b_n/E)$. Let $\alpha(x)$ be the formula:
$(d_q u u_1 \dots u_n)(\exists y_1 \dots y_n)(\bigwedge{}_i D(y_i,u_i) \ \& \ x=f(u,y_1,\dots,y_n))$. Clearly $\alpha(c')\Rightarrow c'/E$ is **P**-internal. In particular, letting E_0 be a set of definition for α, (*) follows.

Preview and explanation It will be seen in §3 that if **P** is in either of the main classes (1) and (2) above, then $A_{\mathbf{P}}=int_{\mathbf{P},\infty}(A)$ can be larger than $int_{\mathbf{P}}(A)$ only in the presence of a definable group. Once one has a group G acting on a set A, one may form a "restricted $\mathbb{C}^{eq}$", consisting of quotients of powers of A under G-invariant equivalence relations. Then the above results remain valid in the restricted context; every sort that needs to be considered is a "G-sort" in a natural way. This in turn can be used to get a definablity result, and then the preceding proposition can be brought into play. This is the main reason for the insistence on the use of definitional rather than algebraic closure here.

proof of Theorem 1

a) Let **P** be the class of all superstable definable sets (considered as partial types.) By hypothesis, a strong type is foreign to **P** if and only if it is algebraic. By Corollary 6, this is equivalent to the statement that every element a is algebraic over $\{a\}_{\mathbf{P}}$; by Proposition 8, $\{a\}_{\mathbf{P}}=\mathrm{int}_{\mathbf{P},\omega}(a)$. So for every a there exists a sequence $a_0,\ldots,a_n$ such that $a_i/\{a_0,\ldots,a_{i-1}\}$ is **P**-internal for each i, and $a \in \mathrm{acl}(a_0,\ldots,a_n)$. But it is easy to see that a strong type is **P**-internal iff it has ordinal R^∞. It does not follow automatically that $R^\infty(a/\emptyset)<\infty$; but it does follow (from the sub-additivity of U-rank) that $U(a/\emptyset)<\infty$. (see [Ls1]). Since a was arbitrary, every type has ordinal U-rank, so T is superstable.

b) Let M be a countable model. Since there are only countably many types with ordinal Morely rank over countable sets (each one is determined by a single formula), there exists a countable elementary extension N of M such that for every finite $A \subset N$, every type with ordinal Morley rank over $M \cup A$ is realized in N. Let **P** be the class of formulas with ordinal Morley rank. By induction on n, it follows that for every integer n and every finite $A \subset N$, every type over $M \cup A$ which is **P**-analyzable in n steps is realized in N. But every type is **P**-analyzable in finitely many steps, as in the proof of (a). So every type over M is realized in N. In particular, there are only countably many types over M. Since M was an arbitrary countable model, T is ω-stable.

Example The cardinality restriction in (2) cannot be avoided. Consider the following theory. The language will have a predicate symbol P_η for each finite sequence η of 0's and 1's, as well as an equivalence relation E_α for each function $\alpha \in \{0,1\}^\omega$. The axioms will say that $(\forall x)P_0(x)$, $P_\eta = P_{\eta\hat{}0} \cup P_{\eta\hat{}1}$, and $P_{\eta\hat{}0} \cap P_{\eta\hat{}1} = \emptyset$ for each η. For each α and each n, an additional axiom will say that $P_{\alpha|n\hat{}(1-\alpha(n))}$ is an equivalence class of E_α. For any element a in a model of T, let α be the unique member of $\{0,1\}^\omega$ that extends each η such that $a \in P_\eta$. Then $tp((a/E_\alpha)/\emptyset)$ has Morley rank 1, multiplicity 1. (As a non-algebraic E_α-class.) The same is true for $stp(a/(a/E_\alpha))$. Thus every type is analyzable (in 2-steps) in terms of types with Morley rank. The theory is clearly not totally transcendental, however.

§2.2 Kueker's Conjecture for stable theories

A Kueker theory is a complete countable first order theory whose every uncountable model is $\aleph_0$-saturated, but that is not $\aleph_0$-categorical. We will show that a stable Kueker theory is $\aleph_1$-categorical. Let T be such a theory. Here are two facts from [Bu1].

1. There are only countably many types over $\emptyset$. (There exist Eherenfeucht-Mostowski models in every cardinality, and they realize only countably many types; but they realize all types.) In particular, T has prime models over finite sets.

2. The prime model over a finite set A is minimal over A. (Otherwise, by a familiar homogeneity argument, there exists an uncountable model

atomic over a finite set; so every type over A is atomic, and T is $\aleph_0$-categorical.)

For the sake of the following lemmas, call a formula D(x) almost complete over A if there exists a unique nonalgebraic complete type p over A with $D \in p$; and call D completable if this type is isolated over A.

Lemma 3 Let M be a prime model over a finite set, let $D \in L(M)$ be a non-algebraic formula, and let $A \subset M$. Then there exists a finite $B \subset M$ (with $A \subset B$) and $D' \in L(B)$ such that $\models D' \rightarrow D$, and D' is almost complete, but not completable, over B.

Proof Write M as the union of an increasing chain of finite sets $A_1 \subset A_2 \subset \ldots$. Let D_0="x=x", and define inductively $D_n \in L(A_n)$ so that $\models D_{n+1} \rightarrow D_n$, D_n is non-algebraic (i.e. has infinitely many solutions), and is almost complete over A_n (i.e. cannot be split into two disjoint non-algebraic formulas over A_n.) This is possible since there are only countably many types over each A_n. Suppose each D_n is completable. Let c be an element such that $\models \bigwedge_n D_n(c)$. Then $c \notin M$, and $tp(c/A_n)$ is isolated for each n. It follows that $M \cup \{c\}$ is atomic, i.e. every n-tuple from $M \cup \{c\}$ realizes an isolated type over $\emptyset$. By the omitting types theorem, there exists an atomic model $N \supset M \cup \{c\}$. N must be prime, contradicting the minimality of prime models. Hence some D_n is almost complete, but not completable.

Lemma 4 Suppose T does not have the finite cover property. Then every almost complete formula is either completable, or weakly minimal.

Proof Without loss of generality T is countable. Let $D(x)$ be almost complete, but not completable, and suppose for convenience that D is 0-definable. Let p be the unique non-algebraic type extending $D(x)$. Suppose $R^{\infty}(D)>1$. Then there exists a type extending p that is not based on $\emptyset$ and is not algebraic. So there exists a formula D_1 such that $D_1 \Rightarrow D$, and

i) D_1 is not algebraic

ii) D_1 causes forking over $\emptyset$

By Fact 1.10, we may choose D_1 to have parameters from D: $D_1 = D_1(x, d_1, \ldots d_n)$ where each $d_i \in D^{\mathbb{C}}$. So we may assume:

iii) $\models (\forall y_1 \ldots y_n)(\forall x)(D_1(x, \bar{y}) \Rightarrow D(x)\ \&\ \bigwedge_i D(y_i))$.

By the nfcp, there exists a formula $\eta_1(\bar{y})$ such that $\models \eta_1(\bar{d}')$ iff $D_1(x, \bar{d}')$ has infinitely many solutions. Since $D_1(x, \bar{d})$ causes forking over $\emptyset$, there is no a such that $a \perp \bar{d}$ and $\models D_1(a, \bar{d})$; in other words, letting $q = stp(a/\emptyset)$, $\models \sim(d_q x) D_1(x, \bar{d})$. Let $\eta_2{}^q = \sim(d_q x) D_1(x, \bar{d})$; then in particular $\models \eta_2{}^q(\bar{d})$ for each strong type q based on $\emptyset$ extending p. Since all these strong types are conjugate over $\emptyset$, the formulas $\eta_2{}^q(\bar{y})$ are also conjugate; since each one is almost over $\emptyset$, there are only finitely many of them. Let $\eta_2(\bar{y}) = \bigwedge\{ \eta_2{}^q(\bar{y}) : q \supset p,\ q \text{ based on } \emptyset\}$. So $\models \eta_1(\bar{d})\ \&\ \eta_2(\bar{d})$.

Let $\bar{d}'$ be such that $\models \eta_1(\bar{d}') \& \eta_2(\bar{d}')$, and $tp(\bar{d}'/\emptyset)$ is isolated. Each component d_i' of $\bar{d}'$ realizes a type extending $D(x)$ which (being isolated) is not equal to p; since D is almost complete, it follows that $d_i' \in acl(\emptyset)$ for each i. Let $D_1' = D_1(x, \bar{d}')$, and let D_1'' be the disjunction of all the (finitely many) conjugates of D_1'. Then $D_1'' \Rightarrow D$, and since $\models \eta_1(\bar{d}')$, D_1'' is non-algebraic. Since D is almost complete, $D - D_1''$ is finite. So if $a \models p$ then $\models D_1''(a)$, and hence $\models D_1(a, \bar{d}'')$ for some conjugate $\bar{d}''$ of $\bar{d}'$; replacing a by a conjugate, we may assume $\models D_1(a, \bar{d}')$. Since $\bar{d}' \in acl(\emptyset)$, $a \perp \bar{d}'$. These two facts contradict the meaning of $\eta_2(\bar{d}')$.

We now show that the results of the preceding section are applicable.

Lemma 5 If p is a finitely based non-algebraic type and D is an infinite definable set, the p is not foreign to D.

Proof Let M be a model, prime over a finite set A, with $D \in L(M)$ and p based on M. Suppose p is foreign to D. By a repeated application of 2.1.2(e), one finds an uncountable model $N \supset M$ with $D^M = D^N$. Since T is a Kueker theory, N is $\aleph_0$-saturated. By lemma (3), there exists a formula $D' \in L(B)$ (some finite $B \subset M$ with $A \subset B$) such that D' is almost complete, but not completable, over B. Let p be the unique non-algebraic type over B with $D' \in p$. $\underline{p}$ has a realization c in N; since $D^M = D^N$, $c \in M$. Since M is prime over A and $A \subset B \subset M$, $p = tp(c/B)$ is isolated. It follows that D is completable, a contradiction.

Conclusion 6 T is superstable.

Proof Choose any non-algebraic type based on $\varnothing$. By the above lemma and 2.1.3, T does not have the finite cover property. Hence by lemmas 3 and 4, there exists a formula D with $R^{\infty}(D)=1$. By lemma 5 and Proposition 2.1.1, T is superstable.

At this point one could have quoted [Bu1], but we will finish the proof.

Claim 7 Let p be a strong type based on a finite set A, and let I be a Morley sequence in p over A, of cardinality $\aleph_1$. Then there exists an atomic model over $A \cup I$.

Proof By lemma 2.3.2 (to follow), it suffices to prove that isolated types are dense over $A \cup I$. Let $\varphi_0 \in L(A \cup I)$ be a consistent formula. There exists a consistent $\varphi(x) \in L(A \cup I_0)$ (some finite, non-empty $I_0 \subset I$) such that (i) $\varphi(x)$ isolates a complete type over $A \cup I_0$, and (ii) no stronger consistent formula over $A \cup I_0$ has smaller R^∞. Since $I_0 \neq \emptyset$, $tp(I/I_0)$ is stationary, so (i) and (ii) imply that $\varphi(x)$ isolates a complete type over $A \cup I$. Thus the isolated types are dense.

Now let $D(x)$ be almost complete but uncompletable over A. Enlarge A if neccessary so that $A \not\subset acl(\emptyset)$. Let p be the non-algebraic type extending D. Choose $a \in A - acl(\emptyset)$, and let $q = stp(a/\emptyset)$. Let I be an uncountable Morley sequence in q over A, and let N be an atomic model over $A \cup I$ provided by the above lemma. Since T is a Kueker theory, p is realized by some $c \in N$. Since $tp(c/A \cup I)$ is isolated, while $p = tp(c/A)$ is not, $c \not\perp I \mid A$. But $R^\infty(p) = 1$; so $c \in acl(A \cup I)$. Now $tp(I/A)$ is stationary; so $p = tp(c/A)$ must have finite multiplicity. Thus there are only finitely many strong types based on A extending p; in other words, only finitely many non-algebraic strong types extending D. This means that D has Morley rank 1. It now follows as in the proof of lemma 6 (using 2.1.1(b) rather than (a)) that T is ω-stable. In particular, all types are finitely based, so lemma 3 says precisely that T has no Vaughtian pairs. By [BL], T is $\aleph_1$-categorical.

§2.3 Kueker's conjecture for theories with a linear ordering.

Proposition 1 Kueker's conjecture is true if one of the following holds:

(a) T interprets an infinite linear ordering.

(b) T has Skolem functions.

(a) generalizes a result of J. Knight, cited in [Bu1], that Kueker's conjecture holds for pure theories of linear orderings.

Proof of 1(b) Let T be a Kueker theory with Skolem functions. Then one may proceed as in the stable case. First note that the following weak version of the nfcp holds:

(*) For any formula $\psi(x,\overline{y})$ there exists an integer N such that for all $\overline{b}$, if $\varphi(x,\overline{b})$ has more that N solutions than it has infinitely many.

For let

$q(u,\overline{z})=\{\sim(\exists x_1...x_n)(\bigwedge_i\psi(x_i,\overline{z})\ \&\ u=f(\overline{x})\ ) : n<\omega,$ f a definable n-ary f'n$\}$

If $\{x\in\mathbf{C}: \varphi(x,\overline{b})\}$ is infinite, let X be an uncountable subset; then the Skolem hull of X is a model, and it omits the type $q(u,\overline{b})$. Since T is a Kueker theory, q must be inconsistent; so there exists a formula $\alpha(u,\overline{x})$ and integers m,n such that $\models(\forall\overline{x})(\alpha(u,\overline{x})$ has at most m solutions), and $\models(\forall u)(\exists x_1...x_n)(\bigwedge_i\psi(x_i,\overline{b})\ \&\ \alpha(u,\overline{x}))$. Write this last formula as $\pi(\overline{b})$. Then it is clear that, conversely, if $\models\pi(\overline{b}')$ then $\varphi(x,\overline{b}')$ must be infinite. We have found a set Π of 0-definable formulas such that $\varphi(x,\overline{b})$ is infinite iff $\models\pi(\overline{b})$ for some $\pi\in\Pi$. (*) follows by compactness.

Next, recall that lemma 3 of the previous section did not use stability; so there exists an almost complete, uncompletable formula D(x) over some finite A. It is now clear that D must be strongly minimal: if it could be split into two definable infinite sets, then by (*) the parameters neccessary for defining these sets could be found in the Skolem hull of A, hence in A; this would contradict the definition of almost completeness

Finally, consider again the argument of the first paragraph. It showed that for any infinite formula D(x), there exist an integer n and n-ary functions $f_1,...,f_m$ such that $\models(\forall x)(x\in\bigcup_i f_i(D^n))$. If D is chosen strongly minimal, this implies that T is almost strongly minimal, and hence $\aleph_1$-categorical.

Proof of 1(a) Let D(x) be an infinite definable set linearly ordered by a definable relation <. We have to show that T cannot be a Kueker theory. Suppose it is. By lemma 2.2.3 we may assume that D(x) is almost complete but uncompleteable over $\emptyset$. Let M be the prime model. Each $a \in D^M$ realizes a type extending D other than the unique non-algebraic type extending D (since the latter is not isolated); so $a \in acl(\emptyset)$. Now the set of conjugates of a is finite and linearly ordered, and Aut($\mathbb{C}$) acts transitively on it, so it must have exactly one element. In other words, $D^M \subset dcl(\emptyset)$. It follows that D cannot be split into two infinite subsets by a formula with parameters from D^M. In particular, every segment of D (under <) is finite or co-finite. It is easy to see that the only linear orderings with this property are $\omega+n$, $n+\omega^*$, and $\omega+\omega^*$ (for $n<\omega$; where ω^* is the opposite ordering to ω.) In each case, one finds that T has Skolem functions inside D. (Given a formula $\varphi(x,\overline{y})$, let $f(\overline{y})$ be the function defined as follows: the domain of f is $\{\overline{y}: (\exists x)(D(x) \,\&\, \varphi(x,\overline{y}))\}$. $f(\overline{y})$ is the least $x \in D$ satisfying $\varphi(x,\overline{y})$ if there is a least such x, otherwise the greatest x with this property.) One would like to apply case (b) to conclude that $T' = Th(D^{\mathbb{C}}$,full induced structure) is $\aleph_1$-categorical; for this we need to show that T' is Kueker.

Lemma 2 Let T be a 1st-order theory, card(T)$\leq\aleph_1$, and suppose the isolated types are dense over $\emptyset$. Then T has an atomic model.

Proof Build the model in $\aleph_1$ steps. At each step one has a countable atomic set, and wants to add an element realizing one more formula. The type realized by the new element can be built in ω steps, and no problem is encountered.

Now let A be model of T' of cardinality $\aleph_1$. A may be considered as a subset of $\mathbb{C}$ = the universal domain of T, in such a way that the structure of A as a model of T' agrees with its structure as a subset of $\mathbb{C}$. (In particular $A \subset D^{\mathbb{C}}$.) Let $T'' = Th(\mathbb{C}, a)_{a \in A}$. I claim that the hypothesis of lemma 2 applies. Indeed, each formula φ (of L) over A extends to an isolated type q over A_0 for some finite $A_0 \subset A$. Since T' has Skolem functions and $A \models T'$, and since the language of T' was chosen to contain the full induced structure, a standard reflection argument shows that q has a unique extension to A. Thus by lemma 2, T'' has an atomic model $(M, a)_{a \in A}$. In particular, it is clear that $D^M = A$. Since T is a Kueker theory, M is $\aleph_0$-saturated, and it follows that A is $\aleph_0$-saturated as a model of T'. Thus T' is itself a Kueker theory.

Applying case (b), we see that T' is $\aleph_0$-categorical or $\aleph_1$-categorical. The first possibility would contradict the fact that T' has Skolem functions (and an infinite model), the second contradicts the existence of a definable linear ordering.

Example In [Ku], Kueker requested an example of a countable model M which is relatively $\aleph_0$-saturated in some proper extension, but it is not relatively $\aleph_0$-saturated in any uncountable extension. The following is a small, superstable example; it can be modified so as to answer some variant questions in the same paper. It also answers a request of Pillay's for a countable superstable model which is homogeneous over $acl(\emptyset)$ but not over $\emptyset$.

Let $M = \dot{\cup}\{{}^n2 : n \leq \omega\}$. (${}^n2$ = set of functions on $\{0,1,\ldots,n-1\}$ into $\{0,1\}$). Let $P_n{}^M = {}^n2$, and define equivalence relations E_n on $M-(P_0{}^M \cup P_1{}^M \cup \ldots \cup P_{n-1}{}^M)$ by: aE_nb iff $a \notin (P_0 \cup P_1 \cup \ldots \cup P_{n-1})$ and $b \notin (P_0 \cup P_1 \cup \ldots \cup P_{n-1})$ and $a|n=b|n$. So E_n has 2^n classes, each one the union of two infinite E_{n+1} classes; and P_n is a complete set of representatives for E_n. $T=Th(M)$ is superstable, with countably many types over $\emptyset$. Let $M_0=acl(\emptyset)=\cup\{{}^n2 : n<\omega\}$, $M_1=M_0 \cup \{a_0,a_1,a_2,\ldots\}$, $M_2=M_0 \cup \{a_0,a_1,a_2,\ldots,a_\omega\}$, where the a_i's are elements of $M-M_0$ chosen so that $a_nE_ma_\omega$ iff $n \geq m$. So $M_0<M_1<M_2<M$. Moreover, M_1 is relatively $\aleph_0$-saturated in M_2. However, there are no other extensions of M_1 with the same property (up to isomorphism over M_1.) To see this note that any small extension N of M_1 must omit the partial type $q(xy)=\{xE_ny \ \& \ x \neq y : n<\omega\}$, so N may be identified with a submodel of M. Moreover for each k, N must omit the type $p_k(x)=\{-P_n(x): n<\omega\} \cup \{-(xE_ka_k)\} \cup \{x \neq a_i : i<k\}$; so every element of $N-M_1$ must be E_k-equivalent to a_ω for every k. a_ω is the only possibility.

§3. Unidimensional Theories

A complete first order theory T is unidimensional if the class of $|T|^+$-saturated models of T is categorical in some large power. Shelah gave a structural characterization of these theories: T is unidimensional if T is stable, and no two nontrivial strong types are orthogonal in a model of T. He asked if such a theory must be superstable. It is shown in §3.4 that this is indeed the case.

Previous work on the problem: Prest and Pillay ([PP]) proved the conjecture for theories of modules (without extra structure.) S. Buechler ([B2]) proved it in the case where T is superstable inside some definable set, using techniques similar to those of chapter 1 (independently of it.)

Once the theory is known to be superstable, a considerable amount of light can be thrown on it. The main fact is Buechler's theorem that such a theory is either totally transcendental, or locally modular. In the second case, the last proposition of §5 shows the existence of an $acl(\emptyset)$-definable Abelian group; by [PH] this group admits elimination of quantifiers to the level of $acl(\emptyset)$-definable cosets of $acl(\emptyset)$-definable subgroups. In view of the excellent co-ordinatization theory available for 1-based theories (Theorem 5.1, Corollary 5.9, and Theorem 4.1), this gives a good feeling for the structure of models of such theories.

The proof of the conjecture involves the theory of stable groups. §3.1 contains a crucial definabity result: a $\bigwedge$-definable group in a stable theory is a subgroup of a definable group. This generalizes a theorem of Poizat's [Pz1]. Somewhat analogously, if not only the underlying set but the group operation itself (on generics) is given by a type rather that a formula, we still find a group presented in the ordinary way. This was originally a part of the proof of Theorem 5.2; Poizat pointed out that it is a model-theoretic version of a result of Weil's, who considered the same structures over algebraically closed fields (with an extra assumption in prime characteristic. See [W].)

§3.2 is a presentation of Shelah's theory of local weight; it answers Question V.4.9 in [Sh]. (It is clear that p-simplicity is the weakest notion making Theorem V.4.1 of [Sh] true; therefore property 3 answers question V.4.1.) The regularity criterion is a bonus, which will be extremely useful for us later on.

In §3.3 it is shown that the theory of local weight goes through very smoothly in the context of stable groups. It provides a generalization of the technology of [BeL]. This was neccessary because the tool of [BeL], U-rank, was not available in the context of §3.4. But it gives a finer resolution even for superstable groups, for two reasons: it allows one to distinguish between two regular types of the same U-rank, and (more importantly) to tell apart semi-regular groups (such as $(Z/2Z)^{\omega} \times (Z/2Z)^{\omega}$, so presented) from ones that are not (such as $(Z/4Z)^{\omega}$).

§3.4 contains the proof that unidimensional theories are superstable.

§3.1. Generically Presented Groups

Theorem 1 Let p be a stationary type, and let * be a definable partial operation such that a*b is defined for generic a,b. Assume:

i) if a,b,c are independent realizations of p, then (a*b)*c=a*(b*c).

ii_L) for generic $a,b \in p^{\mathbf{C}}$, $a*b \models p|a$.

ii_R) for generic $a,b \models p^{\mathbb{C}}$, $a{*}b \models p|b$. (I.e. $a{*}b \perp b$).

Then there exists a formula G, a definable operation $\cdot$ inducing a group structure on G, and a definable embedding of p into G such that for independent a,b, the image of $a{*}b$ is the product of the images of a,b.

proof Some trivialities need to be mentioned before the proof can begin. A germ of a definable function at a stationary type p is its equivalence class under the equivalence relation: f,g are equivalent if they agree generically. More precisely, since f,g are definable they may be considered as objects in $\mathbb{C}^{eq}$. f,g define the same germ if for $a \models p|\{f,g\}$, $f(a)=g(a)$. Note that this equivalence relation is definable (on definable families of definable functions), by the open mapping theorem; so the germ of a definable function f may itself be considered as an object in $\mathbb{C}^{eq}$. The following claim is easy but important:

<u>Claim</u> Let g be a germ and $a \models p|g$. Let f be a definable function with germ $\overline{f}=g$ satisfying $f \perp a$. Then the value of f(a) is independent of the choice of f. (And hence will be written as g(a).)

<u>proof</u> Let f_1,f_2 be two such functions. Choose f_3 with germ g and $f_3 \perp \{f_1,f_2,a\}/g$. Then f_3,f_1,a are independent over g, so $\{f_3,f_1\} \perp a|g$. Since $a \perp g$, $\{f_3,f_1\} \perp a$. Thus by definition of a germ, $f_3(a)=f_1(a)$. Similarly $f_3(a)=f_2(a)$. So $f_1(a)=f_2(a)$.

The claim will be considered as a justification to consider germs of definable functions by themselves, without seperate notation for the function they come from. So from now on f,g will denote germs.

So far we have considered only the domain of germs explicitly. If p,q are stationary types, we may consider the germs mapping p into q, i.e. the set of all germs of definable functions f at p with the property that for

$a \models p | B$, $f(a) \models q | B$ (where B is any set such that f,p, and q are B-definable.) We obtain the structure of a category, whose objects are the paralleslism classes of stationary types, and whose morphisms are the germs of definable functions, with the obvious composition. In particular for each p we have the semi-group with-identity $S(p)$ of germs of functions $p \to p$. Note that $S(p)$ satisfies the right cancellation law: if $g \cdot f = h \cdot f$, then $h=g$. (proof: let $a \models p | \{f,g,h\}$. $a \perp f,g,h$; so $f(a) \perp g,h | f$. But $f(a) \perp f$, so $f(a) \perp g,h$. Thus the fact that $g(f(a))=h(f(a))$ means that h=g generically, i.e. h=g.) The invertible members of this semigroup form a group, denoted $G(p)$, the group of generic permutations of p. $G(p)$ need not be definable, of course.

The plan is to identify each element a with the germ $\overline{a}$ of the map $x \mapsto a*x$, to take G=the semi-group generated by these germs, to show that it is generated in 2 steps and hence Λ-definable, and to observe that it forms a group.

First we must show the "identification" is in fact an embedding, i.e. that $c_1=c_2$ if $\overline{c_1}=\overline{c_2}$. Indeed, pick $b \models p | \{c_1,c_2\}$. So for some a_1,a_2 we have $ai*b=ci$. (This uses (ii_R).) For generic d,

$a_1*(b*d)=(a_1*b)*d=c_1*d=c_2*d=(a_2*b)*d=a_2*(b*d)$. Since $b*d$ is generic to $\{a_1,a_2\}$, this shows that $\overline{a_1}=\overline{a_2}$. So using the fact that the germ suffices to define a function, $c_1=a_1*b=\overline{a_1}(b)=\overline{a_2}(b)=c_2$. Thus we have indeed embedded p into a semigroup. If a,b are independent, then $\overline{a*b}=\overline{a}\cdot\overline{b}$, as is immediate from (ii). So we may truly identify p with a subset of $S(p)$, and $*$ with multiplication $(\cdot)$ in $S(p)$.

To show that the semi-group generated by the elements realizing p is (Λ-) definable comes down to this: for all $a,b,c \models p$ there exist $e,f \models p$ s.t. $a \cdot b \cdot c = e \cdot f$. To see this, pick $b_1 \models p | \{a,b,c\}$. So $b=b_1 \cdot b_2$ for some $b_2 \models p$. (As was argued in the previous paragraph.) b_2 and b_1 are equi-definable over b, so since $b_1 \perp c | b$, also $b_2 \perp c | b$. By assumption (iii), $b_2 \perp b$; so $b_2 \perp c$. Thus letting $e = a \cdot b_1$ and $f = b_2 \cdot c$, we have $a \cdot b \cdot c \models p$ and (since p is closed under generic multiplication) $e, f \models p$. We have embedded p in a $\mathbf{A}$-definable semigroup S_0.

A $\mathbf{A}$-definable semi-group with cancellation in a stable theory is a group: given $a \in S_0$, we need to find $b \in S_0$ such that $ba = 1$. By compactness, what must be shown is that for every definable set $D_0 \supset S_0$, such a b exists inside D_0. Without loss of generality $\cdot$ is defined on on $D_0 \times D_0$ and satisfies the right cancellation law there. Let $D_1 \subset D_0$ be a definable set such that $S_0 \subset D_1$ and $x \cdot y \in D_0$ for $x, y \in D_1$. Write $u | v$ for $(\exists x \in D_1)(x \cdot u = v)$. For $m \leq n < \omega$, $a^m | a^n$. By stability, $a^m | a^n$ for some $m > n$. So there are $n \geq 0$ and $k > 0$ and $c \in D_1$ such that $c \cdot a^{n+k} = a^n$, or $(c \cdot a^{k-1}) a a^n = a^n$. Using right cancellation, $(c \cdot a^{k-1}) \cdot a = 1$. So every element a of S_0 has a left inverse a^{-1} in S_0. Now $a \cdot a^{-1} \cdot a = a$, so by right cancellation again $a \cdot a^{-1} = 1$. Thus S_0 is a group.

It is obvious that p is the generic type of S_0. The remaining point is that S_0 is a subgroup of a formula-definable group. This follows from the next proposition.

Proposition 2 Let P be a partial type, $\cdot$ a definable operation inducing a group structure on P. Then there exists a formula G in P such that G is closed under $\cdot$, and is a supergroup of P.

The proof was inspired by Poizat's theorem, that assuming the conclusion of the proposition, P is the intersection of definable subgroups. (We will later use this fact, which can easily, if inelegantly, be made to follow from the proposition.) By 1.1.11, P must be the interesection of at most $|T|$ definable subgroups.

Proof By §1.1.13, for any formula $\psi(x,y)$ there exists a formula $\rho(x)$ such that for all a, $\models\rho(a)$ iff for every generic type q of P, for $c\models q|a$, $\models\psi(a,c)$.

Let Φ be a set of formulas such that $P=\cap\{\varphi: \varphi\in\Phi\}$. Pick $S_0\in\Phi$ such that $x,y,z\in S_0 \Rightarrow x\cdot(y\cdot z)=(x\cdot y)\cdot z$ and $x\cdot 1=x$. (Compactness.) For any $\varphi\in\Phi$, let $S_\varphi = \{x\in S_0 : \text{for generic } y\in P,\ x\cdot y\in\varphi\}$. Then

(*) $P = \cap\{S_\varphi: \varphi\in\Phi\}$. (If $a\in$ each S_φ then for generic $b\in P$, $a\cdot b\in P$ so (since $a,b\in S_0$) $a=a\cdot 1=a\cdot(b\cdot b^{-1})=(ab)b^{-1}\in P$.)

So there exists $\psi\in\Phi$ such that $S_0\subset\psi$ and:

(**) $x,y\in S_\psi \Rightarrow x\cdot y\in S_0$.

Let $S_1=S_\psi$ for this ψ. In particular, if $a\in S_1$ and $b\in P$ then $a\cdot b\in S_0$. I claim that in fact $a\cdot b\in S_1$. By definition of $S_1=S_\psi$, this means that for generic $c\in P$, $(a\cdot b)\cdot c\in\psi$. Pick such a c; by the properties of generic types, $b\cdot c$ is generic to a. Since $a\in S_1$, $a\cdot(b\cdot c)\in\psi$. But $a,b,c\in S_0$ so $(a\cdot b)\cdot c\in a\cdot(b\cdot c)$.

Let $S_2=\{y\in S_1: (\forall x)(x\in S_1 \Rightarrow x\cdot y\in S_1)\}$. The last paragraph showed precisely that $P\subset S_2$. So S_2 is a definable semigroup containing P. The set of invertible elements of S satisfies the requirements for G.

Everything could have been done for group actions rather than groups.

Definition Let G be a connected $\bigwedge$-definable group, with generic type q. A germ of a definable function $f: q\otimes p\to p$ is called a generic action of G on q if for $(\sigma,\tau,a)\models q\otimes q\otimes p$, $f(\sigma,f(\tau,a))=f(\sigma\tau,a)$. $f(\sigma,a)$ is written $\sigma\cdot a$.

Proposition 2 Let G be a connected $\mathbf{\Lambda}$-definable group with a generic action on a strong type p_0. Let B be a base for p_0 and let $p=p_0|B$. Then there exists a $\mathbf{\Lambda}$-definable set P, a definable action of G on P, and an injection of $p^{\mathbf{C}}$ into P preserving the action of G generically. Identify $p^{\mathbf{C}}$ with its image under this embedding. Then $P=\{\sigma a: a\models p\}$. P is unique up to a (unique) definable G-isomorphism over $p^{\mathbf{C}}$. There exists a definable set $P_0\supset P$, a definable group $G_0\supset G$ and a definable action of G_0 on P_0 such that P is G-invariant and the induced action is the original one.

proof Consider the set of pairs (g,a) with $g\in G, a\models p$. Define an equivalence relation $\sim$ on them by: $(g,a)\sim(g',a')$ iff for generic $h\in G$, $(hg)\cdot a=(hg')\cdot a'$. Denote the class of (g,a) by [g,a], and the set of classes by $\bar{P}$. If $hg_2\cdot a=hg_2'\cdot a'$ holds for generic h, then so does $hg_1g_2\cdot a=hg_1g_2'\cdot a'$, since hg_1 is no less generic than h. Thus one can define an action of G on P by $g_1\cdot[g_2,a]=[g_1g_2,a]$. Embed p into P by $a\mapsto[1,a]$. To prove that $G\cdot p=P$, let [g,a] be any element of P. Let $h\perp g,a$ realize a generic type of G. Then $hg\perp a$, so $h[g,a]=[hg,a]=[1,hg\cdot a]$, or $[g,a]=h^{-1}\cdot[1,hg\cdot a]$. Uniqueness is clear from the nature of the construction. The proof of the last statement is contained in the proof of the previous proposition.

Fact 3 Let σ be a germ, with domain p and range q. Then $stp(\sigma)$ is p-internal.

Proof Let B_0 be any set such that p,q, and σ are defined over B_0. Let I_0 be a long Morley sequence of realizations of $q|B_0$, and let $B=B_0\cup I_0$. Let $I=\{\sigma(c): c\in I_0\}$. Then $\sigma\in dcl(B\cup I)$. For suppose $tp(\sigma'/BI)=tp(\sigma/BI)$. Then in particular, $\sigma(c)=\sigma'(c)$ for each $c\in I_0$. Since I_0 is long, $\{\sigma,\sigma'\}\perp c$ for some $c\in I_0$. So $\sigma(c)=\sigma'(c)$ for generic $c\models q$. By definition of a germ, $\sigma=\sigma'$.

Remark 4 Let an Abelian group A be given. Then everything in this section can be done for A-endomorphisms instead of definable functions. For example, a $\bigwedge$-definable field is an intersection of definable fields. In this context, however, the distinction between germs and functions is inessential: every germ of a homomorphism $A \to B$ extends uniquely to a (total) definable homomorphism. This allows us to consider definable homomorphisms as objects in $\mathbb{C}^{eq}$. (Actually for the last statement, nothing more is required than the fact that two homomorphisms are equal iff they agree generically, so equality between them is a definable relation.) This will be implicitly used in §5.2.

Quotients

The study of groups whose theory is stable quickly became the study of stable groups, i.e., groups interpretible in stable structures. The proofs involved $\bigwedge$-definable groups, so e.g. [Be] switched to studying them. This caused some simplification, but also some awkwardness (even in the superstable context) since the class of $\bigwedge$-definable groups is not closed under quotients. Here is one possible solution. We will not actually adopt it, since in the sequel we will consider only groups that arise naturally from the questions tackled at that moment; but its existence will save us from having to excercise care in statements such as "the generic types of G/N are p-internal".

Abstractly, it is often convenient to consider types in infinitely many variables; call them $*$-types. In §2.1, we could have allowed the families **P** of partial types to consist of partial $*$-types, without making any changes. (The main point to remember is that equality is no longer a definable by a single formula.) A $*$-definable set is the solution set (in the universal domain) of a partial $*$-type. (So an element of such a set is an indexed sequence of elements of $\mathbb{C}^{eq}$.) A $*$-definable function is a function whose graph is a $*$-definable set; by compactness it arises from an indexed sequence of definable functions. Now define a $*$-definable group to be a group whose underlying set is $*$-definable, and whose operations are $*$-definable functions. The results about stable groups summarized in §1 hold for $*$-definable groups just as well as for $\mathbf{A}$-definable ones, with the same proofs.

Our definability result can be generalized to give a canonical form to $*$-definable groups. (In effect the following proposition combines Proposition 2 with its dual.)

Definition A projective system of definable groups is a directed partially ordered set J, of small cardinality, a definable group G_j for $j \in J$, and a definable group homomorphism $h_{j_1,j_2}: G_{j_1} \to G_{j_2}$ for $j_1 \geq j_2 \in J$, forming an inverse system, such $h_{j_1,j_1} = id$ and $h_{j_2,j_3} \circ h_{j_1,j_2} = h_{j_1,j_3}$ when $j_3 \leq j_2 \leq j_1$.

Example It is clear that the projective limit of a projective system of definable groups is always a $*$-definable group. (To be precise: given a prjective system as above, there exists a $*$-definable group G $*$-definable maps $\pi_j: G \to G_j$ such that $(G, \pi_j)_j$ is the projective limit of $(G_j, h_{j_1,j_2})_J$; and $(G, \pi_j)_j$ is unique up to a unique isomorphism which is also $*$-definable.)

This is the only example:

Proposition 5 Let G be a $*$-definable group in a stable structure. Then there exists a projective system of definable groups with projective limit G', such that G is isomorphic to G' by a $*$-definable isomorphism.

Proof Let G be a $*$-definable group. Let I be the index set for the $*$-type defining G, so that an element of G has the form $\overline{b}=(b_i: i\in I)$. Let J be the set of finite subsets of I, and for $\overline{b}\in G$ and $j\in J$ let $\overline{b}(j)=(b_i: i\in j)$. Define equivalence relations E_j on G by: $\overline{b^1}E_j\overline{b^2}$ iff for all generic types q^1,q^2 of G, if $(\overline{c^1},\overline{c^2})\models q^1\otimes q^2\,|\,\{\overline{b^1},\overline{b^2}\}$ then $(\overline{c^1}\cdot\overline{b^1}\cdot\overline{c^2})(j)=(\overline{c^1}\cdot\overline{b^2}\cdot\overline{c^2})(j)$. By compactness, for each $j\in J$ there exists $j^*\in J$ such that for $\overline{a^1},\overline{a^2},\overline{a^3}\in G$, $(\overline{a^1}\cdot\overline{a^2}\cdot\overline{a^3})(j)$ depends only on $\overline{a_1}(j),\overline{a_2}(j),\overline{a_3}(j)$. So if $\overline{b_1}(j^*)=\overline{b_2}(j^*)$ then $\overline{b^1}E_j\overline{b^2}$; and moreover by 1.1.13, there exists a formula δ_j in variables $x^1{}_i,x^2{}_i$ $(i\in j^*)$ such that $\overline{b^1}E_j\overline{b^2}$ iff $\models\delta_j(\overline{b_1}(j^*),\overline{b_2}(j^*))$. It follows that there exists a Λ-definable set G_j and a $*$-definable surjective map $\pi_j:G\to G_j$ whose kernel is E_j. The notation G_j is easily justified by verifying that the E_j-class of 1 is a normal subgroup of G, and E_j is the coset decomposition of G. It is also clear that the maps π_j seperate points on G: if $\overline{b^1}E_j\overline{b^2}$ for all j, pick a generic type q of G and let $\overline{c^1},\overline{c^2}\models q\,|\,\{\overline{b^1}\overline{b^2}\}$. Then by definition $\overline{c^1}\cdot\overline{b^1}\cdot\overline{c^2}(j)=\overline{c^1}\cdot\overline{b^2}\cdot\overline{c^2}(j)$ for each j, so $\overline{c^1}\cdot\overline{b^1}\cdot\overline{c^2}=\overline{c^1}\cdot\overline{b^2}\cdot\overline{c^2}$, and hence $\overline{b^1}=\overline{b^2}$.

By proposition 2, for each j there exists a definable group $\hat{G}_j$ such that G_j is a subgroup of $\hat{G}_j$; so the above argument gives an embedding of G onto a Λ-definable subgroup G' of the ($*$-definable) group $\Pi_j\hat{G}_j$. Let K be the set of all finite subsets J, and let $\hat{G}_k=\Pi_{j\in k}\hat{G}_j$. If $k_1\subset k_2$ where $k_1\in K$ and $k_2\in K\cup\{J\}$, we have the projection $\pi_{k_2,k_1}:\hat{G}_{k_2}\to\hat{G}_{k_1}$. For each k, let $\{D_i:i\in I_k\}$ be a small system of definable subgroups of $\hat{G}_k$, whose intersection is $\pi_{J,k}(G')$. (This uses the remark following the statement of proposition 2.) Let $J^* =\{(k,i): k\in K,\ i\in I_k\}$, and define $(k,i)\le(k',i')$ iff $k\subset k'$ and $\pi_{k',k}(D_{i'})\subset D_i$; if these conditions hold let $\pi_{(k',i'),(k,i)}=\pi_{k',k}|D_{i'}$. It is clear that G' is the projective limit of the system just described.

Corollary (To proof.) The class of *-definable groups is closed under quotients.

Remark The reader may prefer the class of *-definable groups with the property that each element is uniformly definable over a singleton of $\mathfrak{C}$; it is also closed under quotients. (In the superstable context it is better since one can continue to use U-rank.) Then one rules out examples such as the group of units of the ring of power series over a stable field.

§3.2 Local Weight

Fix a regular type p. A stationary type q is said to be hereditarily orthogonal to p if p is foreign to q. This notion is invariant under parallelism in both variables. q is <u>p-simple</u> if for some set B with p,q based on B, there exist $c\models q|B$ and an independent sequence I of realizations of $p|B$ such that stp(c/B∪I) is hereditarily orthogonal to p. The minimal possible cardinality of I is called the p-weight of q, or $w_p(q)$. Note that q is hereditarily orthogonal to p iff q is p-simple and $w_p(q)=0$. All these notions are invariant under parallelism, and do not change if p is replaced by a $\square$-equivalent regular type.

Properties

1. Let stp(a/X) be p-simple. If $X\subset Y$ then stp(a/Y) is p-simple. If $a'\in acl(a)$ then stp(a'/X) is p-simple. In both cases the p-weight is at most that of stp(a/X). p is p-simple. $w_p(p)=1$.

2. If $stp(a/X)$ and $stp(b/X)$ are p-simple, and at least one is $\Doteq$-equivalent to some power of p, then $a \perp b \mid X$ iff $w_p(a/bX)=w_p(a/X)$.

3. (Additivity) If a/X and b/X are p-simple, then so is ab/X, and $w_p(ab/X)=w_p(b/X)+w_p(a/bX)$.

4. (Existence) If $stp(a/X)$ is $\not\perp p$ then there exists $a' \in dcl(Xa)$ such that $stp(a'/X)$ is p-simple, of nonzero weight.

5. (Regularity Criterion.) Let $q=stp(a/X)$ be p-simple, X algebraically closed, $w_p(q)=n$. Then $q \Doteq p^{(n)}$ iff for every $d \in dcl(X \cup \{a\})$, if $w_p(d/X)=0$ then $d \in X$. If n=1 and the criterion holds then q is regular.

proofs

(1)-(3) follow easily from the definitions and the elementary properties of forking and of regular types. (See [Sh,Ch. V])

4. Let $a \perp Y \mid X$, $X \subset Y$, p based on Y, $c \models p \mid Y$, $a \not\perp c \mid Y$. Let $e=Cb(stp(Yc/Xa))$. To understand e, let $Yc, Y_1c_1, Y_2c_2, \ldots$ be a Morley sequence over $X \cup \{a\}$. So $e \subset dcl(Y_1Y_2 \ldots c_1c_2 \ldots)$. Now $Y, Y_1, Y_2, \ldots$ is a Morley sequence in a type based on X, so $a \perp YY_1 \ldots \mid X$. Since $e \in acl(X \cup \{a\})$, $e \perp YY_1 \ldots \mid X$. Thus $stp(e/X)$ is parallel to $stp(e/YY_1 \ldots)$. But over $YY_1 \ldots$, e is definable from $c_1c_2 \ldots$, and $stp(c_i/Y_i)$ is p-simple. By (1), $stp(e/YY_1 \ldots)$ is p-simple. Since $Xa \perp Yc \mid e$ by definition of e, $a \perp c \mid Ye$; but $a \not\perp c \mid Y$; so $e \not\perp c \mid Y$. This shows that $w_p(c/X)>0$. Let d' be a finite part of e with the same property. The only problem is that $d' \in acl(Xa)$ rather that $dcl(Xa)$, and this is easily solved by letting d be an element of $\mathbb{C}^{eq}$ equi-definable with the (finite) set of all conjugates of d' over Xa. (We have used the fact that, since $p \not\perp X=acl(X)$, any conjugate of p over X is $\Doteq p$, and hence the notion of p-simplicity is defined over X.)

5. By definition of p-simplicity and p-weight, there exists $Y \perp a \mid X$, $X \subset Y$, and a set I of n independent realizations of $p \mid Y$, such that stp(a/YI) is hereditarily orthogonal to p. If $a \trianglelefteq I \mid Y$ we are done. Otherwise, there exists C with $C \perp I \mid Y$ and $a \not\perp C \mid Y$. Let C_0=Cb(stp(aI/YC). Since $C_0 \subset$acl(YC), $C_0 \perp I \mid Y$; since $Ia \perp C \mid C_0Y$, $a \perp C \mid C_0Y$, so $a \not\perp C_0 \mid Y$; and since $I \perp C \mid Y$ and stp(a/IY) is hereditarily orthogonal to p, the argument of (4) shows that the same is true of stp(C_0/Y). Now let e=Cb(stp(C_0Y/Xa)). Then again stp(e/X) is parallel to a type which is definable over extensions of conjugates of stp(C_0/Y), hence is hereditarily orthogonal to p. As in e is not in acl(X), and it may be replaced by something in dcl(Xa) with ease.

If n=1, i.e. if q is p-simple and $q \not\perp p$, then by (2) every forking extension of has p-weight 0, hence is $\perp p$, and hence $\perp q$; so q is regular.

Remark 3.1 The notion of p-simplicity will be used essentially axiomatically, the axioms being the above five properties. It can thus be replaced by certain other notions. Assume, for example, that p is strongly non-orthogonal to ∅, i.e. any two conjugates of p are non-orthogonal. Let ≈p denote the class of conjugates of p. So ≈p-internal implies p-simple. Then "p-simple" and "hereditarily orthogonal to p" could be replaced everywhere by "≈p-internal" and "≈p-internal of p-weight 0" without changing the truth value of the propositions. This would make some results weaker, others stronger. The main advantage is that there are fewer p-internal types of p-weight 0; for example if T is superstable, and p has U-rank ω^α for some α, then a ≈p-internal type has p-weight 0 just in case it has U-rank $<\omega^\alpha$. The drawback is that the notion is not invariant under $\boxminus$-equivalence, and would have cramped our notation a little had we adopted it.

§3.3 Groups and Weight

Fix a regular type p.

Definitions

G is p-simple (of weight n) if some generic type of p is p-simple (of weight n).

G is p-semi-regular (of weight n) if it is connected, and its generic type is p-simple and $\boxminus p^n$.

G is regular if it is connected and has a regular generic type; equivalently, if it is semi-regular of weight 1.

G is $\mathbf{P}$-internal if some generic type of G is p-internal.

The five properties of local weight translate as follows.

G1. If G is p-simple, then every type in G^{eq} is p-simple. (i.e. any type of an element definable over some elements of G is.)

G2. If G is p-semi-regular of weight n, and q is any type of elements of G with $w_p(q) \geq n$, then q is the generic type.

G3. (Additivity of weight) The abstract formulation is obviously as strong as possible, but note the special case: $w_p(G/S)+w_p(S)=w_p(G)$ if S is a subgroup of a p-simple group G.

G4. (Existence) Suppose the generic type of G is non-orthogonal to p. Then G has a normal subgroup N such that G/N is p-simple, and the generic of G/N is non-orthogonal to p.

G5. (Regularity Criterion) Let G be p-simple, of weight n. Then G is p-semi-regular iff there is no 0-definable normal subgroup $N \neq G$ such that p is foreign to G/N.

proofs

(G1) is clear, since any two generic types are translates of each other, so all generics are p-simple, and any element of G is definable from generic elements.

(G2) has the same proof as for any other forking-sensitive, invariant measure: Let q be the generic type. If $w_p(a/o)=n$, let $b \models q | a$. Then $b \cdot a \models q | \emptyset$. Also $w_p(b \cdot a/b)=w_p(a/b)=n=w_p(b \cdot a/\emptyset)$, so $b \cdot a \perp b$ by the "abstract" property (2). Thus $a=b^{-1} \cdot (b \cdot a) \models q$.

(G3) Immediate.

(G4,G5) By the corresponding abstract property and the following lemma.

Lemma 6

a) Let G be a group. Suppose the generics of G are not foreign to **P**. Then there exists a relatively definable normal subgroup N of G such that G/N is infinite and **P**-internal.

b) If G acts generically on $q=stp(a/\emptyset)$, and q is not foreign to **P**, then there exists a non-algebraic strong type $\hat{q}$ based on $\emptyset$, a generic action of G on $\hat{q}$, and an $acl(\emptyset)$-definable function h such that $h(a) \models \hat{q}$ if $a \models q$, $\hat{q}$ is **P**-internal, and h is a homomorphism of G-sets.

Proof (a) is easy to reduce to the case of a connected group G; in this case it follows from (b) applied to the action of $G\times G$ on G given by $(g_1,g_2)\cdot g_3=g_1g_2g_3^{-1}$. To prove (b), start with 1.1.5, which gives $\hat{q}_0$ and h_0 such that $\hat{q}_0$ is **P**-internal and non-algebraic, h_0 is acl($\emptyset$)-definable, and $h_0(a)\models\hat{q}_0$. Let p be the generic type of G. Let $g(\sigma,b)=h_0(\sigma\cdot b)$ for $(\sigma,b)\models p\otimes q$, and let g_b be the p-germ of the function $\sigma\to g(\sigma,b)$. Let $\hat{q}=stp(g_a/\emptyset)$ and let $h(a)=g_a$. Then by the properties of generics and by 3.1.3 we have what we want.

(A more detailed proof of (a) is implicit in 3.1.5).

Remark 7 (G5) may be restated as follows:

Let G be p-simple, of weight n. Then G has a (unique) p-semi-regular subgroup of weight n. Call it the p-component of G.

Proof Let N be the intersection of all subgroups N_0 of G such that p is foreign to G/N_0. Then clearly p is foreign to G/N. [This means nothing more than that p is foreign to G/N_0 for every definable $N_0\supset N$]. By (G5) (and transitivity) N is p-semi-regular, and by (G3) it has the right weight. As for uniqueness, if N' is another p-semi-regular subgroup of weight n, then by (G1) and (G3) G/N' is p-simple of weight 0, so p is foreign to G/N', and hence $N\subset N'$. But then N'/N is p-semi-regular of weight 0 and connected, so N=N'.

Corollary 8 Let G be a superstable group. Then there exist an integer n and normal subgroups N_i of G ($i\leq n$), such that $N_0=G$, $N_n=(1)$, $N_i\supset N_{i+1}$, and N_i/N_{i+1} is semi-regular for each i. Each N_i may be taken definable almost over $\emptyset$. (In particular, G has a non-trivial semi-regular subgroup.)

<u>proof</u> The N_i's are obtained inductively; N_{i+1} is obtained by choosing a regular type non-orthogonal to the generics of N_i and using lemma 6 applied to the action of $G \times N_i$ (semi-direct product) on N_i given by $(g,n_1)\cdot n_2 = gn_1n_2g^{-1}$. Eventually one reaches (1) because superstability gives a descending chain condition on connected subgroups.

Corollary 7 A simple stable group whose generic types are non-orthogonal to a regular type p is p-internal and p-semi-regular.

The corollary generalizes Zil'ber's theorem that a simple group of finite Morley rank is unidimensional, which is usually proved using his indecomposability theorem. (Here is a statement of the indecomposability theorem in terms of local weight. Let G be a group, p a regular type. Assume that G is p-semi-regular, or at least that there exists a finite bound on the p-weight of types of elements of G. Let q be p-semi-regular type of elements of G, and let C be the subset of G generated from q^G by the operation $(x,y,z)\mapsto xy^{-1}z$. Then C is $\wedge$-definable and p-semi-regular. We will not use this result.)

<u>Existence of Abelian Subgroups</u> If G is p-semi-regular and has no p-semi-regular subgroups of properly smaller weight, then G is Abelian; the usual Reineke argument proves this. For us, however, the following theorem will suffice: regular groups are Abelian. ([Pz2]).

§3.4 Unidimensional Theories are Superstable

Theorem 1 Every unidimensional theory is superstable.

Theorem 2 Let $\mathbf{R}$ be a collection of partial types over $\emptyset$. Suppose $stp(a/\emptyset)$ is $\mathbf{R}$-internal, but $a \perp \bar{b}$ for all sequences $\bar{b}$ of elements realizing $\mathbf{R}$. Then there exists a $\bigwedge$-definable group G and a definable transitive action of G on $\{c: stp(c/\emptyset)=stp(a/\emptyset)\}$. G acts as a group of automorphisms over $\{c: c \models \mathbf{R}\}$. G and its action are ($\bigwedge$-)definable over $\emptyset$.

(Conversely, it is obvious that the conclusion of the theorem implies its hypothesis.)

Corollary 3 This explains when an element is $\mathbf{R}$-analyzable without being $\mathbf{R}$-internal; for by 2.1.5, if a is $\mathbf{R}$-analyzable in 2 steps, then the hypothesis applies to $stp(a/int_{\mathbf{R}}(a))$. This is true if $\mathbf{R}$ is a set of partial types over $\emptyset$, but also (over $acl(\emptyset)$) if $\mathbf{R}$ is the set of conjugates of a regular type p with p non-orthogonal to $\emptyset$. The second case almost follows from the first (applying it to $\mathbf{R}'=\{q \mid acl(\emptyset): q$ is $\mathbf{R}$-internal and based on $\emptyset\}$, and using 1.1.6), so it will not be proved seperately.

Proof of Theorem 2 Let $\underline{R_0}=\{y: y \models \mathbf{R} \mid \emptyset\}$, $\underline{R}=\{y: \exists y_1,\ldots,y_n \in \underline{R_0}.\ y \in acl(y_1,\ldots,y_n)\}$, $G_R=\{\sigma: \sigma$ is an automorphism of $\mathbf{C}$ fixing $\underline{R}$ pointwise$\}$, $q=stp(a/\emptyset)$, $\underline{Q}=\{c: c \models q \mid \emptyset\}$. We have to show that for each $\sigma \in G_R$, $\sigma \mid \underline{Q}$ coincides with a definable function $f(x,b)$, that f does not depend on σ, and that $\{b: f(x,b) \mid \underline{Q}=\sigma \mid \underline{Q}$ for some $\sigma \in G_R\}$ is $\bigwedge$-definable.

Since q is $\mathbf{R}$-internal, there exists b such that if $a \models q \mid b$ then $a \in dcl(b,d)$ for some $d \in \underline{R}$. So $a=g(b,d)$ for some 0-definable function g. Let

$C=Cb(stp(d/\underline{R}))$. (This is unproblematic despite the fact that $\underline{R}$ is not a small set.) We have $C\subset\underline{R}$ (by 1.1.10, or because $\underline{R}=acl(\underline{R})$), so every $c\in\underline{Q}$ realizes $q|C$. Let $p=stp(b/C)$, $\underline{P}=\{b': b'\models p|C\}$. If $b\in\underline{P}$ and $d\in\underline{R}$ then $b\perp d|C$, so $b\models p|C\cup\{d\}$. It is an easy excercise in stability and saturation that if a_1,a_2 realize the same type over each $b\in\underline{R}$, then there exists $\sigma\in G_R$ with $\sigma(a_1)=a_2$. Hence G_R acts transitively on both $\underline{Q}$ and $\underline{P}$.

Let $\sigma\in G_R$ and $(a,b)\models q\otimes p|C$. Then there exists $d\in\underline{R}$ such that $a=g(b,d)$. Let $r_0=tp(d/\emptyset)$. Applying σ, we get $\sigma a=g(\sigma b,d)$, and in fact we have that for all $d\models r_0$, $\models(a=g(b,d))\equiv(\sigma a=g(\sigma b,d))$. By compactness, there exists a formula $\rho\in r_0$ (depending on $a,b,\sigma a,\sigma b$) such that $\models\varphi_\rho(b,\sigma b,a,\sigma a)$, where $\varphi_\rho(y_0,y_1,x_0,x_1)\equiv(\forall z)(\rho(z)\rightarrow(x_0=g(y_0,z)\equiv x_1=g(y_1,z)))\&(\exists z)(x_0=g(y_0,z))$.

Obviously for any b_0,b_1,a_0 and any $\rho\in r$ there exists at most one a_1 such that $\models\varphi_\rho(b_0,b_1,a_0,a_1)$; on the other hand, if $(a_0,b_0)\models q\otimes p$ and $b_1\models p$ then there exists $\sigma\in G_R$ with $\sigma b_0=b_1$, so there does exist some ρ and some a_1 such that $\models\varphi_\rho(b_0,b_1,a_0,a_1)$. One may conclude using compactness that there exists a fixed ρ such that for all a_0,b_0,b_1 with $(a_0,b_0)\models q\otimes p$ and $b_1\models p$, $\models(\exists x_1)\varphi_\rho(b_0,b_1,a_0,x_1)$; and it follows that for any $\sigma\in G_R$, and any $a_0,b_0\models p\otimes q$, $\models\varphi_\rho(b,\sigma b,a,\sigma a)$ for this fixed ρ. So we have a 0-definable partial function h_0 (defined by $h_0(y_0,y_1,x_0)=x_1$ iff $\models\varphi_\rho(y_0,y_1,x_0,x_1)$) such that $h_0(b,\sigma b,a)=\sigma a$. Let $I=\{b_k: k<\omega\}$ be an independent set over C with $b_0=b$, and $\lambda=card(T)^+$.

We have:

(*) For all k, for $a\models q|C\cup\{b_k\}$, $h_0(b_k,\sigma b_k,a)=\sigma a$

It follows that:

(**) There exists an integer n such that for all $a\models q$ and all σ there exists a subset $F\subset\omega$ of cardinality n such that for $k\notin F$, $h_0(b_k,\sigma b_k,a)=\sigma a$.

For suppose (**) fails. Then by compactness there exists Morley sequences $(c_k: k<\lambda+\lambda)$ and $(d_k: k<\lambda+\lambda)$ over C and $\hat{a}\models q$ such that for all

$k,k'<\lambda+\lambda$, for all $a\models q|C$ such that $a\perp\{c_k,c_{k'}\}$, $h_0(c_k,d_k,a)=h_0(c_{k'},d_{k'},a)$ [by (*),] and if $k<\omega$ and $\omega<k'<\omega+\omega$ then $h_0(c_k,d_k,a)\neq h_0(c_{k'},d_{k'},a)$ [by the negation of (**)]. This is easily seen to be impossible, using 1.8.
So let $\bar{b}=(\bar{b}^0,\bar{b}^1)$ where $\bar{b}^0=(b_0,\ldots,b_{2n})$ and $\bar{b}^1=(\sigma b_0,\ldots,\sigma b_{2n})$, and define a function $h=h_{\bar{b}}$ by:
$h(a)=a'$ iff $\{k<2n: h_0(b_k,\sigma b_k,a)=a'\}$ has cardinality at least $n+1$.
By (**), h is well defined on $\underline{Q}$, and $h(a)=\sigma(a)$ for $a\in\underline{Q}$. It is of course important that $h_{\bar{b}}$ is definable uniformly in $\bar{b}$.

We needed $b_1,\ldots,b_{2n},\sigma b_1,\ldots,\sigma b_{2n}$ to show that $\sigma|\underline{Q}$ is a definable map for each σ, and in particular it is meaningful to speak of the q-germ of σ. But once this is known they can be dispensed with; $h_{\bar{b}}$ actually depends on b_0 and σb_0 alone. It is clear from the construction of h that the q-germ of σ depends only on b_0 and σb_0; so it remains to show that each σ is determined by its q-germ. In other words, if $\sigma\in G_R$, $\sigma|\underline{Q}=h_{\bar{b}}|\underline{Q}$ and σ fixes every $a\in\underline{Q}$ with $a\perp\bar{b}|C$, then σ is the identity on $\underline{Q}$. Let σ be such an element, $\sigma|\underline{Q}=h_{\bar{b}}|\underline{Q}$. Pick $c\in\underline{P}$ with $c\perp\bar{b}|C$. Let $a\models q|C\cup\{\bar{b},c,\sigma c\}$, and let $d\in\underline{R}$ be such that $a=g(c,d)$ and $d\perp\sigma c|C\cup\{a,c\}$. By the choice of a, $\sigma c\perp a|C\cup\{c\}$, so $d\perp\sigma c|C\cup\{c\}$. As $c\perp\underline{R}|C$, $d\perp c|C$, so $d\perp\{c,\sigma c\}|C$. But $a\perp\bar{b}$, so σ fixes a, and hence $g(\sigma c,d)=g(c,d)$. This shows that $g(c,z)$ and $g(\sigma c,z)$ define the same r-germ, where $r=stp(d/C)$. By the Claim of §3.1, $g(c,d)=g(\sigma c,d)$ whenever $d\models r|C\cup\{c\}$ and $d\models r|C\cup\{\sigma c\}$. In the present case, this just means that $g(c,d)=g(\sigma c,d)$ whenever $d\models r$. So $\sigma g(c,d)=g(c,d)$ for all d. As every element of $\underline{Q}$ has the form $g(c,d)$ for some d, σ is indeed the identity on $\underline{Q}$.

We have now shown that for each $\sigma\in G_R$ and each $b\in\underline{P}$, $\sigma|\underline{Q}$ agrees with a definable function $f(x,b,\sigma b)$ with no invisible parameters. (The form of f does not depend on σ by the uniformity of the argument.) It is now easy to recognize the pairs b_1,b_2 such that $f(x,b_1,b_2)$ agrees with some member of G_R on $\underline{Q}$: (b_1,b_2) is such a pair iff there exists C' and p' conjugate to C,p such that $b_1 \models p'|C$ and $b_2 \models p'|C$. This is a Λ-definable set $\underline{S}$. We have also shown that $f(x,b_1,b_2)|\underline{Q}=f(x,b_1',b_2')|\underline{Q}$ iff $f(x,b_1,b_2)$ and $f(x,b_1',b_2')$ have the same q-germs. The latter expression is a definable equivalence relation on $\underline{S}$. The quotient is canonically isomorphic to $\{\sigma|\underline{Q}: \sigma \in G_R\}$, and it is clear that the induced group structure and action on $\underline{Q}$ are definable.

Proof of Theorem 1 Let T be stable and unidimensional. Let us distinguish two cases. Recall that by 3.1.2, if T has a Λ-definable infinite group then it has a definable one.

Case 1 T does not interpret an infinite group.

Let p be a minimal type. Without loss of generality p is based on $\emptyset$. Pick any element a. By 2.1.5 and since T has no non-algebraic orthogonal types, $a \in acl(a')$ where a' is p-analyzable over $\emptyset$. By theorem 2 (or rather corollary 3 to it) and the non-existence of groups, every p-analyzable type is p-internal. It is clear from the definition that if $stp(a'/\emptyset)$ is p-internal than it has finite U-rank. Thus $stp(a/\emptyset)$ has finite U-rank. Since a was arbitrary, T is certainly superstable.

Case 2 There exists a definable group G.

Then the generics of G are not orthogonal to the minimal type p. By 3.3.6, there exists a definable normal subgroup N of G such that H=G/N is p-internal and infinite. Being p-internal, a generic type of H has finite U-rank; since every group element is a product of two generics, every type in

H has finite U-rank. It follows that $R^{\infty}(H)$ is finite. By 2.1(a) T is superstable.

§4. Almost Orthogonal Regular Types

§4.1

Theorem 1 Let $\overline{p,q}$ be non-orthogonal regular types. Let n be the largest integer such that $\overline{p}^{(n)}$ is almost orthogonal to $\overline{q}^{(\omega)}$. Then there exist regular types $p \,{}^{a}\!\sqsubseteq \overline{p}$ and $q \,{}^{a}\!\trianglelefteq \overline{q}^{(\omega)}$ with the following structure:

<u>If n=1</u>, q is the generic type of a $\mathbf{\Lambda}$-definable Abelian group A. There exists a definable action of A on the extension of p, isomorphic to the regular action. p has no structure other than that induced on it by this action from q.

<u>If n=2 or n=3</u>, then q is the generic type of a $\mathbf{\Lambda}$-definable algebraically closed field F. p has the structure of an affine or projective line over F (depending on whether n=2 or n=3); and it has no structure above what is thus induced from F. In particular, any n distinct realizations of p are independent.

<u>n≥4</u> is impossible.

p may be obtained from $\overline{p}$ simply by factoring out an acl(ø)-definable equivalence relation. The group or field mentioned are acl(ø)(-$\mathbf{\Lambda}$)-definable. If T is superstable, then of course the field is definable (and q is strongly regular), and in case 1 the group is the connected component of a definable group.

Theorem 1, Version 2 Let $\overline{p,q}$ be regular types based on ø. Let n be an integer such that $\overline{p}^{(n)} \,{}^{a}\!\perp \overline{q}^{(3)}$ and $\overline{p}^{(n+1)} \,{}^{a}\!\not\perp \overline{q}$. Then there exist regular types $p \,{}^{a}\!\sqsubseteq \overline{p}$ and $q \,{}^{a}\!\sqsubseteq \overline{q}$ such that the same conclusions hold.

Proof

The second version will be proved in the next section. Assume the hypothesis of the first.

A definable automorphism group.

This part of the proof has been presented in advance. Let $\underline{Q}$={$b \in \mathbb{C}$: for some $b_1,...,b_n \models \overline{q}$, stp($b/b_1,...,b_n$) is hereditarily orthogonal to $\overline{p}$}, $\mathbf{Q}$={tp($b/\varnothing$): $b \in \underline{Q}$}. Since $\overline{p},\overline{q}$ are non-orthogonal, $\overline{p}$ is not foreign to $\mathbf{Q}$. By 2.1.6, if $\overline{a} \models \overline{p}$ then there exists $a \in$ dcl($\overline{a}$) with stp($a/\varnothing$) $\mathbf{Q}$-internal and non-algebraic. Let p=stp($a/\varnothing$), and let n be the integer from the statement of the theorem. Let $\underline{P}^n$ be the set of realizations of $p^{(n)}$ (not the Cartesian power.) Then no $a \in \underline{P}^n$ can fork over $\varnothing$ with any independent set of realizations of $\overline{q}$; by the regularity of $\overline{q}$ and the definition of $\underline{Q}$, no realization of $p^{(n)}$ can fork with any $b \in \underline{Q}$. By theorem 3.4.2 there exists a $\mathbf{\Lambda}$-definable group G with a definable action on $\underline{P}^n$, isomorphic to the action of the group of all automorphisms of $\mathbb{C}$ fixing $\underline{Q}$ pointwise (modulo those that fix $\underline{P}$ pointwise.) This action is induced by an action on $\underline{P}$, which we will now study.

Claim G is p-semi-regular, of weight n.

proof Note that the fact that G is transitive on $\underline{P}^n$ implies immediately that the generic types of G have p-weight at least n. We will show that every strong type of G has p-weight at most n, and that equality holds for exactly one strong type. The assertion will follow. Let $a_1,...,a_N$ be elements of $\underline{P}$ with the following property: $n \leq N$, $a_1,...,a_n \models p^n$, and if $\sigma \in G$ fixes $a_0,...,a_N$ then $\sigma=1$. They exist by stability and compactness. Let C=Cb(stp($a_0,...,a_N/\underline{Q}$). By 3.2.5, stp($\overline{a}$/C)$\doteq p^m$ for some m. (For if

$e \in dcl(C \cup \{\bar{a}\})$ and $stp(e/C)$ is hereditarily orthogonal to p then $e \in \underline{Q}$, but $\bar{a} \perp \underline{Q} | C$, so $e \in acl(C)$.) Since $a_1 \ldots a_n \perp \underline{Q}$, $m \geq n$. On the other hand n was chosen maximal, and this implies the other inequality. So $stp(\bar{a}/C) = p^n$. Let $\sigma \in G$ realize any type over C, with $\sigma \perp \{\bar{a}\} | C$. Let $b_i = \sigma a_i$. Then $stp(\bar{b}/C) = stp(\bar{a}/C)$ (as G fixes $acl(C)$). By the choice of $a_1, \ldots, a_N$, σ is the unique member of G satisfying $\sigma\bar{a} = \bar{b}$; so $\sigma \in dcl(\bar{a}, \bar{b})$. Thus $w_p(\sigma/C) = w_p(\sigma/C \cup \{\bar{a}\}) \leq w_p(\bar{b}/C \cup \{\bar{a}\}) \leq n$. Suppose equality holds; then also $w_p(\bar{b}/C \cup \{\bar{a}\}) = n = w_p(\bar{b}/C)$. By 3.2.2, $\bar{a} \perp \bar{b} | C$, so $\overline{ab} \models stp(\bar{a}/C)^2 | C$. Since $\sigma \in dcl(\overline{ab})$ in a fixed way, this leaves only one possibility for $stp(\sigma/C)$.

Sharp Transitivity

Lemma G acts sharply transitively on $\underline{P}^n$.

proof Let $A = (a_1, \ldots, a_{n-1}) \models p^{n-1}$. Let G_A be the set of elements of G that fix each a_i. Let $\underline{P}_A$ be the set of elements of $\underline{P}$ free from A. Then G_A acts transitively on $\underline{P}_A$, so $w_p(G_A) \geq 1$; and it is easy to see that in fact $w_p(G_A) = 1$. By 3.3.7, G_A has a p-semi-regular subgroup H of weight 1; in other words, H is regular. (Actually it will soon be known that G acts sharply transitively on $\underline{P}^n$, and hence $H = G_A$.) By Poizat's theorem, H is Abelian. By weight considerations, H acts transitively on $\underline{P}_A$. (For generic $h \in H$ and $a \in \underline{P}_A$, $ha \perp a$, hence any two independent elements of $\underline{P}_A$ are H-conjugate.) But Abelian groups have only one faithful transitive action, the regular one. So if we pick a generic $\sigma \in H$, it has no fixed points on $\underline{P}_A$. Thus $Fix(\sigma) =_{def} \{c \models p : \sigma c = c\} \subset \{c \models p : c \not\perp A\}$. As $w_p(A/\emptyset) = n-1$, $Fix(\sigma)$ cannot contain n independent points, so σ does not fix any realization of p^n.

Now consider σ as a member of G. Clearly $\sigma \perp a$ for each $a \in A$. Also $w_p(\sigma/A) = 1$. By additivity, $w_p(\sigma/\emptyset) = n$. We have found one member of G, of p-weight n, that does not have n independent fixed points. But the only type in G of p-weight n is the generic type; so no element in G of p-weight n fixes n independent points.

Now let σ be an arbitrary member of G and suppose σ does fix an independent sequence $a_1,\dots,a_n$. By the above $w_p(\sigma)<n$. By definition of p-weight, $\sigma \perp a_i$ for some i. So σ fixes an element generic to it, and it follows that $\sigma=1$. In other words, σ is sharply transitive on P^n.

<u>The opposite group</u>

Whenever a group G acts on a set S, one can consider G^*={permutations of S commuting with each member of G). If G acts sharply transitively on S, then so does G^*, and G^* is anti-isomorphic (hence isomorphic) to G. The anti-isomorphism depends on the choice of an element a of A: once a is chosen, define $h_a{:}G\to G^*$ by: $\sigma^*=h_a(\sigma)$ iff $\sigma^*a=\sigma a$. G^* and its action on S are definable in the model (G,S); e.g. take $G^*=S\times S/$(G-conjugacy). The anti-isomorphisms are also definable, but if G is non-Abelian then parameters are required. (If G is Abelian then all the h_a's coincide.)

Now consider the case of the group G acting on $\underline{P}^n$. Being isomorphic to G, G^* is p-semi-regular, of weight n. If n=1 then $G=G^*$. But if $n\geq 2$ then G and G^* are different as $\mathbf{A}$-definable groups: G clearly cannot be $\subset\underline{Q}$ (It acts on p, and $p^{2a}\perp\underline{Q}$), but G^* is $\mathbf{Q}$-internal. The proof of this statement is simply "Galois theory." Every automorphism of $\mathbb{C}$ that fixes $\underline{Q}$ fixes G^* pointwise (G^* is defined to be the set of maps that commute with each element of $G=\mathrm{Aut}(\underline{P}^n/\underline{Q})$); so every element σ^* of G^* is in $\mathrm{dcl}(\underline{Q})$, hence in $\underline{Q}$. ($\mathbb{C}$ is saturated enough for this.)

Let $A=\{\sigma^* \in G^*$: for generic $\overline{a} \in \underline{P}^n$, if $\overline{a}=(a_1,\dots,a_n)$ and $\sigma^* \cdot \overline{a}=(b_1 \dots b_n)$ then $a_i=b_i$ for $i=1,\dots,n-1\}$. A is a 0-definable subgroup of G^*. Since $G^* \subset \underline{Q}$, any element of G^* is automatically free from any given $\overline{a} \in \underline{P}^n$; so $A=\{\sigma^* \in G^*$: for all $\overline{a} \in S$, $(\sigma^* \cdot \overline{a})_i=a_i$ for $i<n\}$. In other words, the anti-isomorphism $h_{\overline{a}}$ takes A to the subgroup of $G_{a_0,\dots,a_{n-1}}$ of G consisting of the elements that fix $a_0,\dots,a_{n-1}$. This shows that A is a connected Abelian group with regular generic type $\approx p$. Let q be the generic type of A. Then $q^{a} \trianglelefteq \overline{q}^{\,\omega}$. This finishes the case n=1. (In the other cases the opposite group will provide a translation of the field structure, which will at first be neither 0-definable nor **Q**-internal, into one satisfying the requirements. A will be the multiplicative group of the field.)

The case n=2

We know that G acts sharply transitively on independent pairs of p. Let

$I=\{\sigma \in G: \ \sigma^2=1\}$

$N=\{\tau \in G$: for generic $\sigma \in I$, $\tau\sigma \in I\}$.

For $a \models p$, let $G_a=\{\sigma \in G: \sigma(a)=a\}$.

The definition of N is justified by the following claim.

Claim 1 $\sigma \in I \Rightarrow w_p(\sigma) \leq 1$. There exists a unique strong type of involutions with p-weight 1, and it is regular.

proof: Let $\sigma \in I$. There are two cases. If $\sigma a \not\perp a$ for generic a, then $w_p(\sigma)=0$: pick b_1,b_2 such that σ,b_1,b_2 are independent; then $w_p(\sigma)=w_p(\sigma/b_1,b_2)=w_p(\sigma b_1,\sigma b_2/\ b_1,b_2)=0$. If $\sigma a \perp a$ for generic a, pick such an a. By sharp transitivity, σ is the unique member of G that transposes a and σa. stp(σ) is determined by this. It is parallel to stp(σ/a) $\approx$ stp(σa/a) $\approx$ p, so it is regular.

<u>Claim 2</u> N is a connected normal subgroup of G, with a regular generic type $\boxminus p$.

<u>proof</u>: The main point is that N is nontrivial. Let σ,σ' be independent elements of I of weight 1. Let $\tau=\sigma'\sigma$. If we could show that $w_p(\tau)\leq 1$, then (since $w_p(\tau/\sigma')=w_p(\sigma/\sigma')=1$) we would have $\tau\perp\sigma'$. It would follow that $\tau\in N$ (the product of τ with a generic member of I being in I), so $w_p(N)\geq 1$. So we need to know that τ is not generic. In the proof of sharp transitivity, generic elements have been shown to have a fixed point. We will show that the product of two generic involutions does not. For suppose $\sigma\sigma'a=a$. Then $\sigma'a=\sigma a=b$ (say). Since σ,σ' are independent, one of them , say σ, is $\perp a$. Distinguish two cases. If $a\perp b$ then $\sigma=\sigma'=$the unique automorphism permuting a,b, a contradiction. If $a\not\perp b$ then $\sigma a\not\perp a$ for generic $a\perp\sigma$, hence as was shown in the previous claim σ has p-weight 0, again a contradiction. Hence $\tau=\sigma\sigma'$ has no fixed points, and N is nontrivial.

Since everything is p-simple and some type inside N has p-weight 1, any generic of N must have p-weight ≥ 1. So it remains only to show that there is exactly one strong type in N of p-weight ≥ 1, and it is regular. Let $\tau\in N$, $w_p(\tau)\geq 1$. Let $\sigma\in I$ be of weight 1 over τ. Then $\tau\sigma\in I$ and $w_p(\tau\sigma/\sigma)\geq 1$. Claim 1 implies that τ and σ must be two independent realizations of the unique strong type of p-weight 1 of I. The strong type of $\tau=(\tau\sigma)\cdot\sigma$ is determined by this.

<u>Claim 3</u> G_a is connected with regular generic type $\boxminus p$. $G_a\cap N=1$. $NG_a=G$.

proof: By the proof of (1), generic elements of N have no fixed points. So if $\sigma \in G_a \cap N$ then $w_p(\sigma/\emptyset)=0$. But then $\sigma \perp b$ for all $b \in \underline{P}$, so the fact that $\sigma a=a$ implies $\sigma b=b$ for all b, i.e. $\sigma=1$. The rest is immediate.

By Poizat's theorem on groups with generic regular types, N and G_a are commutative. It is already easy to get a (*-definable) field by factoring out the center of G, but we do not want to do this. The following claim refers to the action of G_a on N be conjugation; Z(G) is the center of G.

Claim 4 The action of G_a on N has a unique orbit outside $Z(G) \cap N$.
proof We need to see that if $n \in N - Z(G)$ $w_p({}^{\tau}n/n)=1$ for generic $\tau \in G_a$. Suppose not. Then ${}^{\tau}n \perp \tau \mid n$, i.e. ${}^{\tau}n$ does not depend on τ. So clearly ${}^{\tau}n=\tau$ for generic $\tau \in G_a$. Hence the centralizer of a contains G_a and n is central, contradiction.

Claim 5 Z(G) is trivial.
Proof Let $\sigma \in Z(G)$. Pick $a \perp \sigma$ and let $b=\sigma a$. If a and b are independent, let τ transpose them. Then $\sigma b=\sigma\tau a=\tau\sigma a=\tau b=a$, so σ,τ agree on a,b, so $\sigma=\tau$ is a generic involution. It follows that all involutions are central, which is clearly absurd. So $a \not\perp \sigma a$ for generic a. Thus $stp(\sigma/\emptyset) \perp p$. In particular, if σ has a fixed point then $\sigma=1$.
Now fix a and write $\sigma=\sigma_1\sigma_2$, with $\sigma_1 \in G_a$, $\sigma_2 \in N$. For $\tau \in G_a$, $\sigma_1\sigma_2\tau=\tau\sigma_1\sigma_2=\sigma_1\tau\sigma_2$, so $\sigma_2\tau=\tau\sigma_2$ and $\sigma_2 \in Z(G)$. Hence also $\sigma_1 \in Z(G)$. But σ_1 fixes a point, so $\sigma_1=1$. Thus $\sigma=\sigma_2 \in N$.
For generic $\tau \in G_a$ and $n \in N$, $w_p({}^{\tau}n/n)=1$, so $w_p(({}^{\tau}n)\cdot n/n)=1$. In particular $({}^{\tau}n)\cdot n$ is not central. Let $\sigma \in G_a$ be s.t. ${}^{\sigma}n={}^{\tau}n\cdot n$. Fix σ,τ,n. For any $\rho \in G_a$, we have:

$\sigma\rho_n=\rho\sigma_n=\rho\tau_n\rho_n=\tau\rho_n\rho_n$, i.e. (*) $\sigma_m=\tau_m\cdot m$ where $m=\rho n$.
But as ρ runs through G_a, m runs through all the generics of N. Since $\{m: \sigma_m=\tau_m\cdot m\}$ is a subgroup, it must be N. Thus $\sigma_m=\tau_m\cdot m$ for all $m\in N$. If m is central, this gives $m=m^2$, i.e. m=1.

<u>6</u> N acts sharply transitively on <u>P</u>. (It is Abelian)

<u>7</u> If $n\in N$ and $a\models p$ then $n \perp a$.
Proof: Suppose otherwise. Since each $\tau\in G_a$ extends to an automorphism of $\mathbb{C}$, whose action on G must be conjugation by τ, $^{\tau}n \not\perp a$ for each $\tau\in G_a$. By (4), n must be central, so by (5) n=1, but then $n\perp a$ anyway.

<u>Fact 8</u> Two distinct realizations of p are independent.
<u>proof</u> Pick $a_1 \not\perp a_2 \in \underline{P}$. By the last two points there exists $n\in N$ s.t. $na_1=a_2$, and $n\perp a_1$. So $na_1 \not\perp a_1$ for generic a_1, whence $stp(n/\varnothing)\perp p$. By the definitions of <u>Q</u> and G, $n\in \underline{Q}$ and so n is fixed by G, i.e. n is central. By (5) n=1, i.e. $a_1=a_2$.

There are no further problems in finding the field structure on the subgroup A of the opposite group (plus a formal element 0_A), and the affine structure on <u>P</u>. Pick any two elements $0\neq 1$ in <u>P</u>. By the last two claims, the maps $n\mapsto n\cdot 0$ and $\tau\mapsto\tau\cdot 1$ are bijections of N wih <u>P</u> and of G_0 with <u>P</u>-{0}, respectively. One verifies immediately that they induce the additive and multiplicative groups (respectively) of a field structure on <u>P</u>, whose 0 and 1 are what the notation implies. The anti-isomorphism associated with the element 0 takes G_0 isomorphically to A, and hence induces a field structure on $A \cup \{0_A\}$. This field structure cannot depend on the choice of the elements 0 and 1 in $\underline{P}$, since $p^{2a} \perp \mathbf{Q}$ and $A \subset \underline{Q}$.

We pass to the cases $n>2$.

Lemma Any n distinct element of $\underline{P}$ are independent.

Proof This follows by an easy induction from the case $n=2$.

Lemma $n \leq 3$.

Proof Choose distinct $a_1,\ldots,a_n \in \underline{P}$. Let H be the subgroup of G fixing $a_1,\ldots,a_{n-2}$. The action of H on $\underline{P}-\{a_1,\ldots,a_{n-2}\}$ is exactly what was studied in the case $n=2$. The conclusion was that the action is isomorphic to that of the group of automorphisms of the affine line over a field. In particular, if two points (such as a_{n-1} and a_n) are specified, one has a definable field structure on $\underline{P}-\{a_1,\ldots,a_{n-2}\}$ with $0=a_{n-1}$ and $1=a_n$. Call this field $F_{\overline{a}}$. Let Σ be the symmetric group on the $n-2$ letters $a_1,\ldots,a_{n-2}$. By sharp transitivity, any $\sigma \in \Sigma$ extends uniquely to an element $\tilde{\sigma}$ of G that fixes a_{n-1} and a_n. This gives an embedding of Σ in G. Each $\tilde{\sigma}$ in the range leaves H invariant, and fixes a_{n-1} and a_n; being an automorphism, it must respect the field structure of $F_{\overline{a}}$. (Once H and its action were known, the field structure was defined using a_{n-1} and a_n alone.) This gives an embedding of a finite group in the automorphism group of an algebraically closed field. The only possibilities are $\Sigma=1$, or $|\Sigma|=2$ and the fixed field of Σ is real closed. The second possibility is out, so $|\Sigma|=(n-2)!=1$, and $n\leq 3$.

The case n=3 We will use the following idea of Cherlin and Berline's. Let A be an regular group, written additively. Then every definable subgroup of A is either equal to A or has p-weight 0 (p being the generic type of A). By the proof of Schur's lemma, every definable endomorphism of A is surjective. It follows that the ring of definable endomorphisms of A has no 0-divisiors. (Actually, it embeds into a division ring.) In particular, the factorization $x^2-1=(x-1)(x+1)$ shows that the only endomorphisms of A of order 2 are the identity and $a \mapsto -a$.

To apply this to the present case, fix 3 points of $\underline{P}$ and call them 0,1, and ∞. By the case n=2, there is a definable field structure $F = F_{0,1,\infty}$ on $P^{\mathbb{C}}-\{\infty\}$. The group G_∞ corresponds to the group G there, and $G_{\infty,0}$ corresponds to the group of elements fixing a point. The multiplicative structure on $\underline{P}-\{\infty,0\}$ is induced from the group structure on $G_{\infty,0}$ by the bijection $g \mapsto g \cdot 1$. So the automorphism $\sigma \in G$ satisfying $\sigma(0)=\infty$, $\sigma(\infty)=0$, $\sigma(1)=1$ respects this structure. Since σ^2 fixes 3 points, $\sigma^2=1$. Thus by the previous paragraph $\sigma(x)=x^{-1}$ for $x \in F_{0,1,\infty}-\{0,\infty\}$.

We now have two definable 3-transitive groups acting on $\underline{P}$: G, and the group of all Möbius transformations associated with $F_{0,1,\infty}$. We want to show that they are equal. Note that a 3-transitive subgroup of a sharply 3-transitive group can only be the entire group. Therefore we will be done as soon as we show that each Möbius transformation is a member of G. It is well known that the group of Möbius transformation is generated by the linear transformations $z \to az+b$ together with the map $z \to 1/z$. The former kind are in $G_\infty \subset G$ by the case n=2, and the latter was exhibited explicitly a moment ago. This finishes the proof.

§4.2 Extensions of the theorem.

There are two natural directions of generalization of theorem 1. The first is the complete classification of the possible sets of integers of the form $\{(m,n): p^{(m)}a\perp q^{(n)}\}$ where p,q are regular types, and the situations in which each such set occurs. (The theorem gives us full information about m if we relinquish all control of n, and vice versa. The second version gives somewhat more precise information, but for example it is not known whether there are infinitely many possibilities for the sets of integers described above.) The second direction is to continue to consider $q^{(\omega)}$ on the right, but to abandon the assumption that p is regular. (It may as well be semi-regular; and it makes no difference if one assumes that q is regular or not.) We have very little information in either direction at present.

The second direction turned out to be equivalent to the study of simple superstable groups. This is obvious with the current presentation of the proof of theorem 1. Let us mention two examples of this equivalence; first stating theorem 1 in terms of superstable groups, and then stating the "next step" in the analysis of simple superstable groups in terms of almost orthogonality to a normal set.

Theorem 1' If G is a simple superstable group, (hence p-semi-regular of weight n for some p,n), and if G has a subgroup of weight n-1, then n=3, G≈PGL(2,F) for some algebraically closed field F, and the action of G on the coset space is isomorphic to the action of PGL(2,F) on the projective line.

Of course, this is only equivalent to theorem 1 modulo theorem 3.4.2, which was originally the main part of theorem 1. Superstability can be replaced with: the generic types of G are non-orthgonal to some regular type. Theorem 1' generalizes the known results of Cherlin and Berline

([Ch1],[ChSh1],[Be1] prove this for $n \leq 3$), and gives a uniform proof. It can be proved by modifying slightly the proof of theorem 1, or it can be deduced from its statement using the following bit of stable abstract nonsense. (In particular, it follows from the following lemma and theorem 1 that a semi-regular group of weight n acting transitively on a weight 1 set must act sharply n-transitively.)

Definition M' is a mild reduction of M if M and M' have the same universe, M' is a reduct of M, and there exist a finite number of elements $c_1,\dots,c_\lambda$ of M' such that every 0-definable set in M is $(c_1,\dots,c_\lambda)$-definable in M'.

Representation Lemma 2 Let G be a group of permutations of a set X, and suppose $M=(G,X,\text{action, extra structure})$ is stable. Then there exists a mild reduction M' of M such that:

(i) Every $\sigma \in G$ (considered as a permutation of X) extends to an automorphism of M'.

(ii) There exists a 0-definable set $\mathbf{O}$ in M'^{eq} such that $G=\{\sigma|X : \sigma \text{ an } M'\text{-automorphism fixing } \mathbf{O} \text{ pointwise}\}$.

(ii) actually follows from (i).

Proof First asume (i) holds for M, and let us show that (ii) follows automatically. It is clear that every member of G can be extended uniquely to an automorphism of M^{eq}, with which it will be identified. Let $\langle p_i : i \in I\rangle$ be a list of all the types of X (over $\emptyset$), each one repeated $|T|^+$ times, and consider the partial type: $Q(\sigma,x_i\ (i \in I)) = \{\sigma \in G,\ \sigma \neq 1\} \cup \{(x_i : i \in I) \models \otimes_{i \in I} p_i\} \cup \{\sigma x_i = x_i : i \in I\}$. If Q were consistent, there would by realizations σ, a_i $(i \in I)$. For each type p of X, there will be some i such that $a_i \models p_i$ and $a_i \perp \sigma$. So $\sigma \in G$ fixes each element of X independent from it. By the claim below, $\sigma=1$. Thus Q is inconsistent. By compactness, there exists an integer λ and types $p_1,\ldots,p_\lambda$ such that if $a_1,\ldots,a_\lambda \models p_1 \otimes \ldots \otimes p_\lambda$ and σ fixes each a_i then $\sigma=1$. Consider the action of G on $X^{\lambda+1}$. Let **O** be the set of orbits. For each element $\sigma \in G$, the unique automorphism σ of M^{eq} agreeing with σ on X is constant on **O**. Thus G may be considered as a set of **O**-automorphisms of X. It remains to show that there are no others. If τ is an automorphism fixing **O** pointwise, let $a_1,\ldots,a_\lambda \models p_1 \otimes \ldots \otimes p_\lambda$. Since τ fixes orbits, there exists $\sigma \in G$ that agrees with τ on $a_1,\ldots,a_\lambda$. But by the choice of λ, each element $b \in A$ is completely determined by the orbit of $a_1,a_2,\ldots,a_\lambda,b$. (If b' is such that $a_1,a_2,\ldots,a_\lambda,b'$ is in the same orbit, then there exists $\sigma \in G$ such that $\sigma a_i=a_i$ and $\sigma b=b'$; but we must have $\sigma=1$, i.e. $b=b'$). Since the automorphism $\rho=\sigma^{-1}\cdot\tau$ fixes $a_1,\ldots,a_\lambda$ as well as this orbit, it fixes b. b was arbitrary, so $\rho=1$. This shows that in fact G is the full group of automorphisms of X over **O**.

<u>Claim</u> Let G act on X faithfully. Let $K=\{b:$ for each $x \in X$ generic over b, $bx=x\}$. Then $K=\{1\}$.

<u>Proof</u> It is easy to reduce to the transitive case, so assume G acts transitively. Let p be a generic strong type of X with respect to this action. Let H be the group of invertible germs of definable functions $p \to p$. We have a homomorphism $G \to H$ given by sending an element of G to its p-germ. $K \subset K'$=the kernel of this map. K' is normal in G, so $\mathrm{Fix}(K')=\{x \in X:$ for all $k \in K'$, $kx=x\}$ is G-invariant. By definition, $\mathrm{Fix}(K') \supset (p|\emptyset)^{\mathfrak{C}}$. The only G-invariant subset of X containing the extension of $p|\emptyset$ is X; so K' fixes X, i.e. $K'=(1)$. Thus $K=(1)$.

Since the claim does not use the assumption that G acts as a group of automorphisms of X, the definitions of λ and hence of $\mathbf{O}$ do not depend on this assumption. Let M' be the model with the same universe as M in each sort, and with the following structure: on G, the group structure alone; on $G\times G\times X$, the graph of the action; on $\mathbf{O}$, the full structure induced from M; on $X^{\lambda+1}\times\mathbf{O}$, the projection; and no other structure. Then every $\sigma\in G$ extends to an automorphism of M' by acting by conjugation on G, and as the constant function on $\mathbf{O}$; and it is clear that every definable set in M is definable in M' from any $a_1,\ldots,a_\lambda\models p_1\otimes\ldots\otimes p_\lambda$.

In order to state the existence problem of "bad" groups in terms of almost orthogonality, let us make the following definition.

Definition Let q be a regular type, p a q-simple type based on $\emptyset$. Define the deficiency function $d=d(p;q)$ as follows. Let $\underline{Q}=\{b\in\mathbb{C}:$ for some $b_1,\ldots,b_n\models q$, $stp(b/b_1,\ldots,b_n)$ is hereditarily orthogonal to $q\}$. Given an integer n, let $\bar{a}\models p^n$, let C be a set such that $\bar{a}\perp\underline{Q}\,|\,C$ and $C\subset\underline{Q}$. Let $d(n)=w_q(\overline{a}/C)$. (This does not depend on the choices made.)

$d(p;q)$ is a non-negative, non-decreasing, eventually constant integral function.

Equivalence 3: (i) and (ii) are equivalent.

i) There are no (strongly minimal) p,q such that $d=d(p;q)$ satisfies $d(1)=2$ and $d(2)=3$.

ii) The only simple groups of semi-regular weight 3 (Morley rank 3) are PGL(2,K) for K an algebraically closed field. (I.e. there are no bad groups.)

The proof is easy at this point, and is left to the reader.

We now pass to the second direction. There are no known examples of almost orthogonality between regular types, other than those encountered in theorem 1. If any other example exists, then by adding parameters it is easy to get the following situation: p,q are regular types, $p\,{}^{a}\!\perp q$, $p^{2}\,{}^{a}\!\boxminus q^{2}$. We will prove that this cannot occur if p is locally modular (§5), or if p,q are strongly regular and not ${}^{a}\!\boxminus$ to a strictly regular type. Call a strong type p based on $\emptyset$ isolated if $p\,|\,acl(\emptyset)$ is isolated.

Lemma 4 Let p is strongly regular, and $r \boxminus p^{(N)}$. If $r\,{}^{a}\!\not\perp p^{(n+1)}$ but $r\,{}^{a}\!\perp p^{(n)}$, then $p^{(n)}$ is isolated. If r is also isolated, then $p^{(n+1)}$ is isolated.

Proof Let $a_1 \ldots a_{n+1} \models p^{(n+1)}$, $c \models r$, $a_1 \ldots a_{n+1} \not\perp c$. Let $\theta(x_1 \ldots x_{n+1} y)$ be the responsible formula; so $b_1 \ldots b_{n+1} \not\perp d$ whenever $\models \theta(b_1 \ldots b_{n+1} d)$ and $tp(d)=tp(c)$. Say p(x) is strongly regular via $\psi(x)$. Let $\pi(x_1 \ldots x_n)=$ $(\bigwedge_i \psi(x_i))\ \&\ (d_r y)(\exists x_{n+1})(\psi(x_{n+1})\ \&\ \theta(x_1 \ldots x_{n+1} y)$. Suppose $\models \pi(b_1 \ldots b_n)$. Let $d \models r\,|\,\{b_1, \ldots, b_n\}$. Let b_{n+1} be such that $\models \psi(b_{n+1})\ \&\ \theta(b_1 \ldots b_n b_{n+1}, d)$. We have $b_1 \ldots b_{n+1} \not\perp d$. It follows that $b_1 \ldots b_{n+1} \models p^{(n+1)}$: otherwise, let I be a maximal subset of $\{1, \ldots, n+1\}$ such that $\{b_i : i \in I\} \models p^{(card(I))}$. Then $card(I)<n+1$, so $\{b_i : i \in I\} \perp d$ by the almost orthogonality assumption. By strong regularity, $stp(b_1 \ldots b_{n+1} / \{b_i : i \in I\}) \perp p$, so by transitivity $b_1 \ldots b_{n+1} \perp d$, a contradiction. Thus $b_1 \ldots b_{n+1} \models p^{(n+1)}$. In particular, $b_1 \ldots b_n \models p^{(n)}$. So π isolates $p^{(n)}$. If r is also isolated, say by ρ, then $p^{(n+1)}$ is isolated by $\pi(x_1 \ldots x_n)\ \&\ \psi(x_{n+1})\ \&\ (\exists y)(\rho(y)\ \&\ \theta(x_1 \ldots x_{n+1} y))$.

Corollary 5 If p,q are strongly regular, $p^2 \boxminus q^2$ but $p\,{}^{a}\!\perp q$ then $p^2\,|\,acl(\emptyset)$ and $q^2\,|\,acl(\emptyset)$ are isolated. Hence $p\,{}^{a}\!\boxminus \hat{p}$ for some strictly regular $\hat{p}$. (obtained by factoring out the definable equivalence relation of forking.)

The following strengthening of the corollary can be used to discourage attempts to interpret an example of $p^2 \boxminus q^2$, $p^a \perp q$ over a field. The strong regularity assumption may also be motivated by theorem 1; the type q there is strongly regular if T is superstable and $n \geq 2$, and one is interested in the relation between q and $\overline{q}$.

Proposition 6 Let p,q,r be non-orthogonal strongly regular types based on $\emptyset$. Assume $p^a \perp q$, $p^a \trianglelefteq q^{(\omega)}$ and $q^a \trianglelefteq p^{(\omega)}$. Then r is isolated.

Proof Let n be the largest integer such that $p^a \perp q^{(n+1)}$. Let $B \models q^{(n)}$. Then $p^a \perp q|B$, $p^a \trianglelefteq q^{(\omega)}|B$ and $q^a \trianglelefteq p^{(\omega)}|B$; and if $r|B$ is isolated then certainly r is isolated. So we may work over B, i.e. we may assume $p^a \trianglelefteq q^{(2)}$. Similarly, we may assume $q \trianglelefteq p^{(2)}$. By the lemma, p^2 and q^2 are isolated. So $\underline{P}$ = the extension of p and $\underline{Q}$=the extension of q are definable sets. Moreover, $\not\perp$ between two elements is a definable equivalence relation on both $\underline{P}$ and $\underline{Q}$, so by factoring it out we may assume it is the identity. It follows, in particular, that each $x \in \underline{P}$ is definable from some number of elements of $\underline{Q}$, and vice versa. (If $tp(x_1/\underline{Q})=tp(x_2/\underline{Q})$, then $x_1 \not\perp x_2$, so $x_1=x_2$.) Hence $Aut(\underline{Q})$ and $Aut(\underline{P})$ may be canonically identified as a single group $\overline{G}$. Let $\underline{R}$={a: there exist $d_1,\dots,d_n \models r$ such that $stp(a/d_1 \dots d_n)$ is hereditarily orthogonal to p}, $G=\{\sigma \in \overline{G}: \sigma a = a \text{ for } a \in \underline{R}\}$. As in theorem 1, G is a Λ-definable, p-semi-regular group, acting faithfully on each of $\underline{P}$ and $\underline{Q}$.

<u>case 0</u> w(G)=0, i.e. G=1.

Then $\underline{P}\subset\underline{R}$ and $\underline{Q}\subset\underline{R}$, so $p^a \trianglelefteq r^{(n)}$ and $q^a \trianglelefteq r^{(m)}$ for some n,m. Since $p^a \perp q$, $p^a \perp r$ or $q^a \perp r$. Hence by the lemma r is isolated.

<u>case 1</u> w(G)=1.

Then G is $\underline{R}$-internal, and it acts sharply transitively on each of P and Q. Let $a_1,a_2 \models p^2$. So $a_1=\sigma a_2$ for some $\sigma\in G$. $a_1a_2 \not\perp b$ for some $b\in\underline{Q}$. So $a_1b \not\perp \sigma$. Thus $p\otimes q^a \not\perp r^{(n)}$ for some n. An argument similar to the lemma will show that for the least such n, $r^{(n)}$ is isolated.

<u>cases 2 and 3</u> w(G)=2 or 3.

By theorem 1, $\underline{P}$ has the structure of the affine or projective line over some definable field K, and G=AGL(1,K) or G=PGL(2,K). It is not hard to see that every non-normal p-semi-regular subgroup S of G of weight w(G)-1 has the form $G_a=\{g\in G: ga=a\}$ for some $a\in\underline{P}$. (If S fixes no point a, then it must be transitive on $\underline{P}$. As $w(S)\leq 2$, the proof of theorem 1 gives a normal subgroup N of S of weight 1 consisting of elements with no fixed points. If G=PGL(2,K) there are no such elements. In the other case it must be that N=S={all elements without fixed points}, so S is normal.) Hence for every $b\in\underline{Q}$ there exists a unique $a\in\underline{P}$ such that $G_a=G_b$. This contradicts the fact that $p^a \perp q$.

By Theorem 1, there are no other cases.

Finally, here is a proof of the second version of Theorem 1. It rules out phenomena such as $p^a \not\perp q^4$, $q^a \not\perp p^4$, $p^a \perp q^3$.

Theorem 1, Version 2 Let $\overline{p,q}$ be regular types based on $\varnothing$. Let n be an integer such that $\overline{p}^{(n)}\ {}^{a}{\perp}\ \overline{q}^{(3)}$ and $\overline{p}^{(n+1)}\ {}^{a}{\not\perp}\ \overline{q}$. Then there exist regular types $p\ {}^{a}{\sqsubset\!\sqsupset}\ \overline{p}$ and $q\ {}^{a}{\sqsubset\!\sqsupset}\ \overline{q}$ such that the conclusions of theorem 1 hold.

Proof Our task is to find a group acting transitively on p^n (for the right p); the fact that it is definable can then be proved as before, and then one can appeal to lemma 2 and quote the original version of the theorem. Let $\overline{a}=a_0\ldots a_n \models \overline{p}^{(n+1)}$, $b\models\overline{q}$, $\overline{a}\not\perp b$. Let $B=Cb(b/\overline{a})$. I claim that $stp(B/\varnothing)$ is regular and ${}^{a}{\sqsubset\!\sqsupset}\ \overline{q}$. For let $\{b_0,b_1,b_2,\ldots\}$ be an independent set over a of elements realizing $tp(b/acl(a))$. Then $B\subset dcl(\{b_0,b_1,\ldots\})$. Suppose $b_i \perp b_0$ for some $i\neq 0$. Then $n+1=w_p(\overline{a})+2\cdot 0=w_p(\overline{a}b_0b_i)=w_p(\overline{a}/b_0b_1)+2\geq w_p(a_0\ldots a_{n-1}b_0b_1)=n+2$. (The last equality by the assumption because $\overline{p}^{(n)}\ {}^{a}{\perp}\ \overline{q}^{(2)}$.) The contradiction shows that $w_p(b_i/b_0)=0$ for each i, so $w_p(B/b_0)=0$. Now $a_0\ldots a_{n-1}\perp b_0$ (as $p^{(n)}\ {}^{a}{\perp}\ \overline{q}$) and $stp(a_0\ldots a_{n-1}/b_0)\ {\sqsubset\!\sqsupset}\ \overline{p}^{\,n}$, so $a_0\ldots a_{n-1}\perp B \mid \{b_0\}$, and by transitivity $a_0,\ldots,a_{n-1}\perp B$. So $stp(B/\varnothing)=stp(B/\{a_0,\ldots,a_{n-1}\})$. The latter is regular, however, because $B\subset acl(\{a_0,\ldots,a_{n-1}\}\cup\{a_n\})$ and $stp(a_n/\{a_0,\ldots,a_{n-1}\})=\overline{p}$ is regular. Thus $stp(B/\varnothing)$ is regular. It is ${}^{a}{\sqsubset\!\sqsupset}\ \overline{q}$ because $B\not\perp b$. Choose $b'\in B-acl(\varnothing)$, and let b'' be the (finite) s et of conjugates of b' over $\{\{a_0,\ldots,a_n\}\}$. (I.e. $b''=\{\sigma b' : \sigma\in Aut(\mathbb{C})$ and σ leaves $\{a_0,\ldots,a_n\}$ invariant.) Then $stp(b''/\varnothing)$ is also regular and ${}^{a}{\sqsubset\!\sqsupset}\ \overline{q}$, so we may replace $\overline{q}$ by it. Since $b''\in dcl(\{\{a_0,\ldots,a_n\}\})$, $b''=f(a_0,\ldots,a_n)$ for some symmetric 0-definable function f. Define an equivalence relation E on the set of realizations of $\overline{p}$ by: $xEy \equiv (d_{\overline{p}}u_1)(d_{\overline{p}}u_2)\ldots(d_{\overline{p}}u_n)(F(x,\overline{u})=F(y,\overline{u}))$, in other words aEb iff the functions $F(a,\overline{u})$ and $F(b,\overline{u})$ have the same $p^{(n)}$-germ. Using the fact that F is symmetric, one sees that if $a_0,\ldots,a_n\models\overline{p}^{(n+1)}$ and $a_0',\ldots,a_n'\models\overline{p}^{(n+1)}$ and a_iEa_i' for each i, then $F(\overline{a})=F(\overline{a'})$. (This does not seem to follow from the Claim of theorem 3.1.1, but the same proof will work.) In particular, if $\overline{a}\models\overline{p}\mid\varnothing$ then $a/E\notin acl(\varnothing)$. Let $p=stp((\overline{a}/E)/\varnothing)$. If

$a_0,\dots,a_n \models p^{(n+1)}$, let $f(a_0,\dots,a_n)$ denote the common value of $f(\overline{a}_0,\dots,\overline{a}_n)$ where $\overline{a}_0,\dots,\overline{a}_n \models \overline{p}$, and $a_i=\overline{a}_i/E$. Let $E(a_0,\dots,a_n)=\{c \in acl(a_0,\dots,a_n): w_p(c/f(a_0,\dots a_n)=0\}$. It is easy to find symmetric functions f_i $(i<I_0)$ such that $E(a_0,\dots,a_n)=acl(\{f_i(a_0,\dots,a_n)\}: i<I_0)$. Let $F(x_0,\dots,x_n)=(f_i(x_0,\dots,x_n): i<I_0)$, and let $q^*=stp(F(a_0,\dots,a_n)/\emptyset)$. By the same argument as before, q^* is regular and $^a \square q$. By 3.2.5, $stp(a_0\dots a_n/F(a_0\dots a_n)) \square p^n$.

<u>Claim</u> Let $a_0,\dots,a_n \models p^{(n)}$ and let $b_\nu \models p \mid \{a_0,\dots,a_n\}$ for $\nu=1,2$. Let $\overline{a}^{(i,\nu)}$ be the n+1 tuple whose j'th entry is a_j if $i \neq j$, and whose i'th entry is b_ν. If $F(\overline{a}^{(i,1)})=F(\overline{a}^{(i,2)})$ for $i=0,\dots,n$ then $b_1=b_2$.

<u>Proof</u> By the definition of identity for realizations of p, this amounts to showing that $\{b_1,b_2\} \perp \{a_0,\dots,a_{n-1}\}$. Let $A=\{a_0,\dots,a_n\}$ and let $A_i=A-\{a_i\}$. We clearly have: $b_1 \perp b_2 \mid A_i$ for each i. Thus $w_p(b_1b_2/A_i)=1=w_p(b_1b_2/A)$. Since $stp(A/A_i)$ is regular, it follows by 3.2.2 that $b_1b_2 \perp A \mid A_i$. Thus $Cb(b_1b_2/A) \subset acl(A_i)$ for each i. Since A is an independent set, it follows that $Cb(b_1b_2/A) \subset acl(\cap_i A_i)=acl(\emptyset)$. Thus in fact $\{b_0,b_1\} \perp \{a_0,\dots,a_n\}$.

Let $G=\{\sigma: \sigma$ is a permutation of $p^{\mathfrak{C}}$ and $F(a_0,\dots,a_n)=F(\sigma a_0,\dots,\sigma a_n)$ for all $a_0,\dots,a_n \models p^{n+1}$. We have to show that G acts transitively on the realizations of $p^{(n)}$. The idea is implicit in the above claim; we will use uniqueness to prove existence, and show that in fact for any $(a_0,\dots,a_n) \models p^{(n+1)}$, G is transitive on the extension of $tp(a_0\dots a_n/acl(F(a_0,\dots,a_n)))$. Quantitatively, we need the following.

<u>Claim</u> Let $a_1,\dots,a_{n+3} \models p^{n+3}$, and let $E=\{f(\overline{a}'): \overline{a}'$ is an n+1-tuple of distinct elements from among $\{a_1,\dots,a_{n+3}\}\}$. Then $w_p(E)=3$.

Notation If $\overline{e}=(e_i: i\in I)$ and $s\subset I$, then $\overline{e}_s=(e_i: i\in s)$. $n=\{0,\ldots,n-1\}$. $[X]^j$=the set of j-element subsets of X.

Proof of claim Let $S=[n+2]^{(n+1)}$. For $s\in S$, let $c_s=F(\overline{a}_s)$. For i=0,1 or 2, let s_i be the subset $\{0,\ldots,n-1\}\cup\{n+i\}$. Consider $T=\{s\in S: c_s \trianglelefteq \{c_{s_0},c_{s_1},c_{s_2}\}\}$. I claim that whenever t has n-1 elements and $t\cap\{\alpha_1,\alpha_2,\alpha_3\}=\emptyset$, if $t\cup\{\alpha_1,\alpha_2\}$ and $t\cup\{\alpha_1,\alpha_3\}$ are both in T then so is $t\cup\{\alpha_2,\alpha_3\}$. Indeed, suppose this is false. Then $C=\{c_{t\cup\{\alpha_1,\alpha_2\}},c_{t\cup\{\alpha_1,\alpha_3\}},c_{t\cup\{\alpha_2,\alpha_3\}}\}$ is an independent set. Since $p^{(n)}\,a_{\perp}\overline{q}^{(3)}$, $C\perp\{a_i: i\in t\cup\{\alpha_1\}\}$. It follows that $w_p(\{a_i: i\in t\cup\{\alpha_1,\alpha_2,\alpha_3\}\})\geq n+3$, a contradiction. It is easy to see that the only subset of S containing s_0,s_1 and s_2 and closed under the operation described above is S itself. Thus T=S, so $w_p(E)=3$.

Corollary Let $a_0,\ldots,a_{n+2}\models p^{(n+3)}$, and $b_0,\ldots,b_n\models p^{(n+1)}$. Suppose $F(a_0\ldots a_n)=F(b_0\ldots b_n)=c$ (say) and $stp(a_0\ldots a_n/c)=stp(b_0\ldots b_n/c)$. Then there exist b_{n+1} and b_{n+2} such that for all $s\in[n+3]^{n+1}$, $F(\overline{a}_s)=F(\overline{b}_s)=c_s$ (say), and $stp(\overline{b}/\{c_s: s\in[n+3]^{n+1}\})=stp(\overline{a}/\{c_s: s\in[n+3]^{n+1}\})$.

Proof

Let $c_s=F(\overline{a}_s)$ and let $E=\{c_s: s\in[n+3]^{n+1}\}$. By the claim, $w_p(E)=3$. By almost orthogonality, $\{a_0,\ldots,a_{n-1}\}\perp E$. Hence $\{a_0,\ldots,a_{n-1}\}\perp E\,|\,\{c\}$. So $w_p(a_0\ldots a_n/E\cup\{c\})=n=w_p(a_0\ldots a_n/\{c\})$. Since $stp(a_0\ldots a_n/c)\dashv p^{(n)}$, it follows from 3.2.2 that $a_0\ldots a_n\perp E\,|\,c$. Similarly $b_0\ldots b_n\perp E\,|\,c$. Thus there exists an automorphism τ of $\mathbb{C}$ over acl(E) such that $\tau(a_i)=b_i$ for $i\leq n$. Let $b_{n+1}=\tau(a_{n+1})$, $b_{n+2}=\tau(a_{n+2})$.

Now fix $c\models q^*$, and fix a type r over acl(c) such that $r=stp(\overline{a}/c)$ for some $a\models p^{(n+1)}$ with $f(\overline{a})=c$. Given $\overline{a}$ and $\overline{b}$ realizing r, define an invertible p-germ of a definable map $p\to p$: let $x\mapsto y$ if $x\perp\overline{a}$ and $y\perp\overline{b}$ and

$F(x,\overline{a_s})=F(y,\overline{b_s})$ for all $s\subset\{0,\dots,n\}$ of cardinality n. Such a y exists by the corollary, and is unique by the first claim.

Letting $\sigma((x_i:i<l))=(\sigma(x_i):i<l)$, it remains to prove:

<u>Claim</u> If $\overline{c}\perp\overline{ab}$ and $\sigma_{\overline{ab}}(\overline{c})=\overline{d}$ then $\sigma_{\overline{cd}}=\sigma_{\overline{ab}}$ generically, and $\sigma_{\overline{cd}}(\overline{a})=\overline{b}$.

<u>Proof</u> Let $a_{n+1+i}=c_i$, $b_{n+1+i}=d_i=\sigma_{\overline{ab}}(a_{n+1+i})$. Let $\overline{c}^{(k)}=(a_k,\dots,a_{k+n})$, $\overline{d}^{(k)}=(b_k,\dots,b_{k+n})$. So $\overline{c}^{(0)}=\overline{a}$, $\overline{c}^{(n+1)}=\overline{c}$, $\overline{d}^{(0)}=\overline{b}$, $\overline{d}^{(n+1)}=\overline{d}$. It suffices to prove that $\sigma_{\overline{c}(k)\overline{d}(k)}=\sigma_{\overline{c}(k+1)\overline{d}(k+1)}$ (generically) for each k. By induction, what must be shown is that given $a_0,\dots,a_{n+1}$ and $b_0,\dots,b_{n+1}$ such that $F(\overline{a_s})=F(\overline{b_s})$ for every $s<[n+1]^n$, letting $s_1=\{0,\dots,n\}$ and $s_2=\{0,\dots,n-1,n+1\}$, $\sigma_{\overline{a}(s_1),\overline{b}(s_1)}=\sigma_{\overline{a}(s_2),\overline{b}(s_2)}$ generically. Opening up the definition of σ, this means that for generic $a_{n+2}\models p$ there exists $b_{n+2}\models p$ such that for all $s<[n+3]^n$, $F(\overline{a_s})=F(\overline{b_s})$. This was proved in the corollary.

Now there is no problem seeing that $\sigma_{\overline{ab}}$ extends to an element of G that takes $\overline{a}$ to $\overline{b}$.

Chapter 5: Locally Modular Regular Types

A regular type is called locally modular if some localization of the associated geometry is modular; a superstable theory is locally modular if every regular type is. There are two known existence theorems: a regular type is locally modular if its geometry is locally finite ([Z] or [CHL]) or if it is weakly but not strongly minimal ([Bu3]). Locally modular theories of finite rank have consequently been well studied, paradigmatically in [CHL].

This chapter is an attempt to give a systematic treatment of the locally modular regular types. In the first section we prove what we can "abstractly," i.e. without utilizing the existence of a definable group. This includes generalizations of the known finite-rank theory. The results of the second section are new even in the finite rank case (or even for strongly minimal sets, as long as they are not locally finite. The locally finite, finite rank case was worked out independently and with greater precision in [Lo].) Let p be a non-trivial locally modular type. We show that the vector space structure promised abstractly by the "fundamental theorem of projective geometry" is in fact model-theoretically present, the underlying Abelian group being definable. This gives a representative of the regular type on which forking can be readily analyzed. The resulting information is then fed back into the study of the original type p, giving a full structure theorem for its geometry. In particular, it is shown that there exists a weight-one type over the original base set whose geometry is outright modular.

The main influence on the second section is Zil'bers paper [Z]. The analysis of forking in a locally modular group is a generalization of [PH] (but that paper has roots in Zil'ber as well.) The first section generalizes the first part of [CHL], using Shelah's theory of local weight; intermediate generalizations have been obtained (earlier) in [Bu4] and (independently) in [P2].

§5.1 Abstract Properties

A dependence relation is called modular if it satisfies the dimension law: $\dim A+\dim B = \dim(A\cup B)+\dim(A\cap B)$ for closed sets A,B; it is locally modular if the equation holds whenever $\dim(A\cap B)\neq 0$. A type p stationary over A is regular if forking is a dependence relation on $\{b\in\mathbf{C} : b\models p|A\}$; it is (locally) modular if, in addition, this dependence relation is (locally) modular. (It is more natural to define a regular type p to be locally modular if there exists a base set B such that the geometry associated with $p|B$ is modular. The next proposition will show that this definition is equivalent.) We will see in the end that the geometry just described is very well behaved; indeed it is isomorphic to an affine or projective geometry over a division ring. But to see this we will have to work first with richer geometries of imaginary elements; these can be shown directly to be outright modular. If p is a regular type non-orthogonal to B, let $\mathbf{D}(p,B)=\{b\in\mathbf{C}^{eq}$: stp(b/B) is p-simple.$\}$ Define a geometry on $\mathbf{D}(p,B)$ using a closure operator: let the p-closure of X, or $\mathbf{Cl}_p(X)=\{b\in\mathbf{D}(p,B)$: $w_p(b/X\cup B)=0\}$. $\mathbf{D}(p,B)$ is called modular if $w_p(X/B)+w_p(Y/B)= w_p(X\cup Y/B)+ w_p(X\cap Y/B)$ for all p-closed sets X,Y (containing the base B.) We note briefly that p-closed sets generally have the same cardinality as $\mathbf{C}$; this may create occasional trivial clashes with standard conventions, whose resolution will be left to the reader.

The following proposition shows, in particular, that local modularity is invariant under parallelism and under domination-equivalence.

Proposition 1 Let p be a regular type. Then the following conditions are equivalent:

1) For some B such that p is based on B, $p|B$ is modular.
2) For all B such that p is based on p, $p|B$ is locally modular.
3) For all B such that $p \not\perp B$, $\mathbf{D}(p,B)$ is modular.

Proof (2)⇒(1) is clear. We will show that (1)⇒(2)⇒(3). Suppose first (3) fails while (1) holds. So $w_p(X/B)+w_p(Y/B)>w_p(X\cup Y/B)+w_p(X\cap Y/B)$ for some p-closed sets X,Y containing B. The weights are of course finite. Equivalently, we have: (*) $w_p(X/X\cap Y)+w_p(Y/X\cap Y)>w_p(X\cup Y/X\cap Y)$. From now on work in a very large saturated elementary extension of $\mathbb{C}$; X,Y are no longer p-closed there, but they are of course relatively p-closed inside $acl(X\cup Y)$, which is all we shall use. Let M be an a-prime model over $X\cap Y$. We may assume $XY \perp M|X\cap Y$. So (*) remains true with M in place of $X\cap Y$. By (1), $p|M$ is modular; so there exists $d\models p|M$ such that $d\not\perp X|M$ and $d\not\perp Y|M$. The problem is to carry d down to the $X\cap Y$. Let $d'=Cb(stp(Md/XY))$. I claim that $stp(d'/X)$ and $stp(d'/Y)$ are both hereditarily orthogonal to p. Let $M_1d_1,M_2d_2,\ldots$ be a Morley sequence over XY with $M_1d_1=Md$. Then $M_1,M_2,\ldots$ is a Morley sequence over XY and $M_1\perp XY|X\cap Y$, so $XY\perp M_1\cup M_2\cup\ldots|X\cap Y$. Since $d'\in acl(XY)$, $d'\perp M_1M_2\ldots$(over X). Thus $stp(d'/X)$ is parallel to $stp(d'/XM_1M_2\ldots)$. But $d'\in dcl(M_1d_1M_2d_2\ldots)$, and $stp(d_i/M_iX)$ is hereditarily orthogonal to p for each i. (It is hereditarily orthogonal to the regular type $stp(d_i/M_i)$, which is $\Leftrightarrow p$.) This shows that $stp(d'/X)$ is hereditarily orthogonal to p, and the dual proof works for Y. Since X,Y are p-closed in $acl(X \cup Y)$ it follows that $d' \in X \cap Y$, so $Md \perp XY \mid X \cap Y$. This gives $d \perp XY$(over M), a contradiction.

It remains to prove that $(3) \Rightarrow (2)$. This is immediate from the corollary to the lemma below.

Lemma 2 Let q,r,s be types over $B=acl(B)$, with r,s regular, $\boxminus p$. Assume (3) holds. If $q \not\perp^a r^{(2)} \otimes s \mid B$ then $q \not\perp^a r^{(2)} \mid B$ or $q \not\perp^a r \otimes s \mid B$.

Proof Without loss of generality q is p-simple. Let $a \models q$, $b_1 b_2 c \models r \otimes r \otimes s$, $a \not\perp b_1 b_2 c$. If $a \not\perp b_1$ then $q \not\perp^a r \mid B$. Otherwise $ab_1 \not\perp b_2 c$. By (3) (and using property (2) of p-weight) there exists e such that $stp(e/B)$ is p-simple, of nonzero weight, and $w_p(e/B \cup \{a,b_1\}) = w_p(e/B \cup \{b_2,c\}) = 0$. Let $B_1 = \{x \in acl(Be)$: $w_p(x/B)=0.\}$ Note that by the regularity criterion $stp(e/B_1)$ is regular. Also $b_1 b_2 c \perp B_1 \mid B$, and therefore the forking relations still hold over B_1: $ab_1 \not\perp b_2 c$, $e \not\perp ab_1$, and $e \not\perp b_2 c$ (all over B_1. The second relation is true since $w_p(e/B_1 ab_1) = 0 \neq w_p(e/B_1)$). If $e \perp a \mid B_1 b_1$ or $e \perp c \mid B_1 b_2$ then $e \not\perp b_1 \mid B_1$ or $e \not\perp b_2 \mid B_1$, so b_1, b_2, c or b_2, b_1, a are dependent over B_1 and hence over B, and therefore $q \not\perp^a r^2 \mid B$. Otherwise, let σ be an automorphism of $\mathfrak{C}$ fixing B_1 and e and such that $\sigma b_2 = b_1$. Then $e \not\perp \sigma c \mid B_1 b_1$, while still $e \not\perp a \mid B_1 b_1$. By regularity of $stp(e/B_1)$, $a \not\perp \sigma c \mid B_1 b_1$, so $a \not\perp \{\sigma c, b_1\} \mid B_1$. Now $stp(\sigma c, b_1/B)$ is parallel to $p^{(2)}$, and $stp(B_1/B \cup \{a\})$ is (hereditarily) orthogonal to p; thus if $\sigma c, b_1 \perp a \mid B$ then $\sigma c, b_1 \perp B_1 \mid B \cup \{a\}$ and hence $\sigma c, b_1 \perp a \mid B_1$, contradiction. Hence $a \not\perp \{b_1, \sigma c\} \mid B$ and $q \not\perp^a r \otimes s \mid B$.

Corollary 3 Assume (3), let p be based on $B=acl(B)$, let X be a non-empty set of realizations of $p \mid B$, and let q be based on X. If $q \not\perp^a p^{(\omega)} \mid BX$ then $q \not\perp^a p \mid BX$.

Proof Let I be a maximal independent subset of X. Let $c \models q | X$. By replacing q with $\mathrm{stp}(cX/BI)$ if neccessary, we see that we may assume $I=X$. Since there exists $c_1 \in \mathrm{acl}(BIc)$ such that $\mathrm{stp}(c_1/BI) \,{}^{a}\!\not\perp p^{(\omega)}$ and is p-simple, we may assume this true of c. By successive applications of the lemma, $\mathrm{stp}(c/BI) \,{}^{a}\!\not\perp p^{(2)}$. Let $a_1a_2 \models p^{(2)} | BI$, $c \not\perp a_1a_2 | BI$. So $cI \not\perp a_1a_2 | B$. By (3), there exists e such that $\mathrm{stp}(e/B)$ is p-simple of nonzero weight, but $w_p(e/Ba_1a_2)=w_p(e/BIc)=0$. If $w_p(e/B)=2$ then $w_p(a_1a_2/BIc)=0$ so $c \not\perp a_1 | BI$, and we are done. So $w_p(e/B)=1$. Since $I \perp a_1a_2 | B$ and $e \not\perp a_1a_2 | B$, it follows that $e \perp I | B$. By the lemma again, $\mathrm{stp}(e/B) \,{}^{a}\!\not\perp p^{(2)}$. Since I contains at least one realization of $p|B$, $\mathrm{stp}(e/BI) \,{}^{a}\!\not\perp p$. Let $a \models p | BI$, $e \not\perp a | BI$. then $w_p(a/BIc) \le w_p(e/BIc) + w_p(a/BIce) = 0+0=0$, so $a \not\perp c | BI$. Hence $q \,{}^{a}\!\not\perp p | BI$. Since $I=X$ we are done.

The following theorem generalizes the co-ordinatization theorem of [CHL]. The examples below show that p-closure cannot be replaced with algebraic closure.

Theorem 5.1 Let p be a locally modular regular type. If $p \not\perp \mathrm{stp}(a/B)$ then there exists $a_1 \in \mathbf{Cl}_p(Ba)$ such that $q=\mathrm{stp}(a_1/\mathbf{Cl}_p(B))$ is regular, $\not\perp p$.

Proof Using the "existence" property of local weight, we may assume $\mathrm{stp}(a/B)$ is p-simple. If $w_p(a/B)>1$ then there exists an extension $\hat{B}$ of B such that $0<w_p(a/\hat{B})<w_p(a/B)$. It is easy to see that $\mathrm{stp}(\hat{B}/B)$ may also be taken p-simple (if neccessary, replace it by $Cb(\mathrm{stp}(a/\hat{B})) \cup B$). By (3), there exists a' such that $\mathrm{stp}(a'/B)$ is p-simple of nonzero p-weight, and $w_p(a'/aB)=w_p(a'/\hat{B})=0$. If $w_p(a'/B)=w_p(a/B)$ then $w_p(a/a'B)=0$ so

$w_p(a/\hat{B})=0$, contradiction. So $0<w_p(a'/B)<w_p(a/B)$. By induction there exists a_1 that works for a'; it clearly also works for a.

Example 1 (There need not be a regular type $\Box p$ over acl(B).)

Consider the following theory. A,I are disjoint infinite sets. $\pi:A\to I$ is a surjection. For $i\in I$, A_i is defined to be $\pi^{-1}(\{i\})$. There is an operation making A_i into an Abelian group; as such it is non-trivial, divisible and torsion free, i.e. a Q-space. If $i,j\in I$, then there is an i,j-definable group isomorphism $\sigma_{ij}:A_i\to A_j$. $\sigma_{ii}=\sigma_{ij}\cdot\sigma_{ji}=id$. The next set of axioms quantify universally over elements $i\in I$ and finite subsets $F\subset I-\{i\}$. Fixing i and F, let $\beta_{jk}=\sigma_{ki}\cdot\sigma_{jk}\cdot\sigma_{ij}$. Linearly order F in some way. Then the subring of $End(A_i)$ generated by $\{\beta_{jk}:j,k\in F,\ j<k\}$ is the free commutative polynomial ring on these generators; moreover, for every nonzero polynomial $p(X)\in Q[X]=Q[x_{jk}:j<k\in F]$, $p(\overline{\beta})$ is an automorphism of A_i. For later convenience, let K_i denote the field $Q(\beta_{kl}:k,l\in I-\{i\})$ The reader can check that the above is a consistent, complete theory.

In fact the theory is ω-stable, 2-dimensional. One dimension, I, is trivial; the other, p, corresponding to any V_i, is locally modular. Each V_i is actually modular. But some $i\in I$ must be fixed for understanding the V_i's. For suppose $a\in\mathbb{C}^{eq}$ and $stp(a/\emptyset)$ is regular, $\Box p$. We will get a contradiction. By regularity, $a\perp i$ for each i, and in fact $a\perp\{i:i\in I\}$. Pick $i_0\in I$. Using modularity, it is easy to see that there must be some $v_0\in V_{i_0}$ such that $v_0\not\perp a|i_0$. Let S be an infinite Morley sequence in $stp(v_0/a)$, $S\perp v_0|a$. For $v\in S$, let $v'=\sigma_{\pi(v),i_0}(v)$. Since S is an indiscernible set over $\{a,i_0\}$, so is $S'=\{v':v\in S\}$. Also $w_p(v'/\{a,i\})\le w_p(v/\{a,i\})=0$ for each i. Thus the elements of S' are pairwise dependent over i; this means that they are linearly dependent over K_i. Remembering the nature of K_i and the indiscernibility of S', there will be some rational form $r[X] \in Q[X]$ such that for any $x \neq y \in S'$ there exists a sequence $\overline{\beta}^{xy}$ of generators of K_i such that $x = r(\overline{\beta}^{xy})y$. So $r(\overline{\beta}^{xy}) \cdot r(\overline{\beta}^{yz}) = r(\overline{\beta}^{xz})$ for distinct x,y,z. Using the fact that the β_{jk}'s are free in K_1 subject to the sole relations $\beta_{jk}\beta_{kj}=1$, it is easy to see that $r=1$. So $\mathrm{card}(S')=1$. This means that if $v \models \mathrm{stp}(v_0/a)$ and $v \perp v_0 | a$ then $v_0 = \sigma_{\pi(v), i_0}(v)$. Let v_0, v_1, v_2 be a Morley sequence over a, $i_1 = \pi(v_1)$, $i_2 = \pi(v_2)$. Then $v_0 = \sigma_{i_1, i_0}(v_1) = \sigma_{i_1, i_0}(\sigma_{i_2, i_1}(v_2)) = \sigma_{i_1, i_0}(\sigma_{i_2, i_1}(\sigma_{i_0, i_2}(v_0)))$, i.e. $v_0 = \beta_{i_2, i_1} \cdot v_0$, or $\beta_{i_2, i_1} = 1$. This contradiction shows that in fact $\Rightarrow p$ has no regular representative over $\emptyset$.

Remark 5 In some sense the regular element does always exist, only it fails to enter $\mathbb{C}^{eq}$. If a locally modular regular type p is non-orthogonal to B, then we will see in the next section that there exists a regular, modular q based on some $u \in \mathbf{Cl}_p(B)$. If $a \models q | Bu$, then p-forking is an equivalence relation on the conjugates of au over B, and the set of equivalence classes, had it existed, would have been the regular representative over B.

Example 2 (where the realization of the regular type cannot be found in $\mathrm{acl}(B \cup a)$, even if B is a saturated model.) Let k be an algebraically closed field, V an infinite dimensional vector space over k, B the set of all 2-dimensional subspaces of V. Let $T = \mathrm{Th}(k,+,\cdot,0,1; V,+,0; \cdot ; B; \in)$ where the second $\cdot$ denotes the action of k on V, and $\in$ is the membership relation between V and B. T is again ω-stable, 2-dimensional. The top dimension, that of the generic type p of V, is locally modular. An element of B is definable in a pair of elements of V, hence is p-simple and $\Rightarrow p^{(2)}$. But it is not difficult to see that there can be no realization of a weight

1- type algebraic over a generic $S \in B$, even over an a-saturated model. (If one existed, it would fork with some $v \in V$. As $v \not\perp S$, one can take $v \in S$. But there exists an automorphism σ strongly fixing S such that $\sigma v \perp v$.)

The following proposition will be useful in the analysis of locally modular groups.

Proposition 6 Suppose p is locally modular, a,b are p-simple, and stp(a/b) and stp(b/a) are both $\sqsubseteq$(powers of p). Then $a \perp b | C$ where $C = acl(a) \cap acl(b)$. In this case the "fundamental weight inequality" holds: $w_p(a/b) + w_p(b) \geq w_p(a)$. (Because $w_p(a/C) + w_p(C) = w_p(a)$.)

proof Find e such that $w_p(e/a) = w_p(e/b) = 0$ and $w_p(a/e) + w_p(b/e) = w_p(ab/e)$. Since a/b is $\sqsubseteq$(a power of p), $a \perp e | b$. Similarly $b \perp e | a$. So $Cb(e/ab) \subset acl(a) \cap acl(b) = C$. Thus $w_p(a/C) + w_p(b/C) = w_p(ab/C)$. By the second property of p-simple types, $a \perp b | C$.

Problem 1 If T is superstable, p is a regular type, and p is *not* locally modular, does the $\sqsubseteq$-class of p have a strongly regular representative? This is the only part of the finite rank theory ([Bu3]) that does not follow from the results proved in this section.

§5.2. Finding the group

All groups are Λ-definable. Only p-simple groups will be dealt with. To analyze a regular group A, the following notation will be convenient. Let $A_0 = A_0(\emptyset) = \{a \in A: stp(a/\emptyset)$ is orthogonal to the generic type of $A\}$. An endomorphism σ of A/A_0 is called definable if $\sigma = \{(x+A_0, y+A_0): y+S = \sigma'(x)\}$

for some relatively definable subgroup S of A and some definable endomorphism σ':A→A/S. Note that stp(σ') is automatically r-simple. If stp($\sigma'/\emptyset$) is orthogonal to the generic type of A, σ is called small. The collection of small endomorphisms of A/A_0 forms a ring $D(A)=D_\emptyset(A)$. By a version of Schur's lemma, it is a division ring. So A/A_0 is a vector space over D. We will see later that if the regular r is locally modular, then all endomorphisms of A/A_0 are small, and D is independent of the base.

Theorem 5.2

(a) Let T be a stable theory, p a non-trivial regular type. Assume p is locally modular. Then there exists a $\boldsymbol{\wedge}$-definable Abelian group (in T^{eq}) whose generic type is regular and domination-equivalent to p.

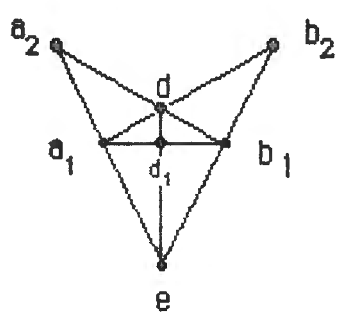
The proof will give a group definable over a set of parameters of p-weight 1. More will be said about this later.

(b) Let A be a $\boldsymbol{\wedge}$-definable Abelian group with regular generic type p, and assume p is locally modular. Let A_0, D be as above, and let $\overline{a}=a+A_0$ for $a \in A$. Let $a_1,\dots,a_n$ be realizations of p. Then $a_1,\dots,a_n$ are independent (in the sense of forking) if and only if $\overline{a}_1,\dots,\overline{a}_n$ are linearly independent over D.

This gives the structure of forking on a certain complete type. For some purposes it is useful to understand forking on a definable set. This can be described as follows. Assume for simplicity that T is superstable, and let p be locally modular. Then there exists a definable Abelian group A, such that any generic type of A is regular and $\bowtie$p. Let D be the division ring described above, for A°. Then there exists a certain vector space V over D, and a map $a \mapsto \bar{a}$ on A into V, with kernel $A_0=\{x \in A: stp(a/\emptyset) \perp p\}$, such that an n-tuple $a_1,\dots,a_n$ of non-orthogonal elements of A is independent (in the sense of forking) if and only if $\bar{a}_1,\dots\bar{a}_n$ are linearly independent. V is a definable direct limit of definable structures. In short, the same picture holds, the difference being that the range of the map $a \mapsto \bar{a}$ is not all of V, but rather a subgroup $\bar{A}$ of V. $\bar{A}$ may not be closed under the action of D, but $\alpha\bar{A} \cap \bar{A}$ will have finite index in $\bar{A}$ for each $\alpha \in D-(0)$. The details will not be carried out here.

proof of part (a). The first part of the proof consists of finding strong types p_1,p_2, both regular and $\not\perp p$, and an invertible germ σ of a definable function from p_1 into p_2 such that $stp(\sigma)$ is again regular, $\not\perp p$. (i.e. finding one group element.) We will silently adjoin parameters several in order to achieve this, in several steps; the notions of forking, regular types, etc. are taken to be over the current base set at each point.

Consider the following diagram:



This is the Zil'ber diagram of [Z], with two additional lines. Each node will stand for a realization of a (possibly infinitary) regular type $\not\perp p$. Two distinct points will be independent; three points are independent iff they are noncollinear; and the total dimension is 3. For a start, choose a base set above which p is modular and non-trivial. By non-triviality, there exist three pairwise independent realizations e, a_1, a_2 of p such that $\{a_1, a_2, e\}$ is not independent. By an argument to be given momentarily, e can be replaced in such a way that $stp(a_1a_2/e)$ is regular, $\doteq p$. Choose b_1, b_2 so that $\langle a_1, a_2 \rangle$ and $\langle b_1 b_2 \rangle$ form a Morley sequence over $\{e\}$. Note that $\{a_1, a_2, b_1, b_2, e\}$ spans a 3-dimensional set in our geometry. Since $\{a_1, b_2\}$ and $\{b_1, a_2\}$ are each independent, local modularity gives $d \models p$ such that $d \not\perp$ each. Similarly one gets d_1 so that each of the five drawn lines is dependent. The reader can check that the 21 pairs and 30 non-colinear triples are indeed independent. In the sequel, two kinds of changes will be made, with a view to getting $d, d_1 \in dcl(e, a_1, a_2, b_1, b_2)$ without losing the other properties. The base set B will be increased to B', where B' will satisfy: $stp(B'/B)$ is hereditarily orthogonal to p. By the regularity of each node, dependence and independence will remain the same over the larger base set. Secondly, elements will be replaced by $\doteq$-equivalent ones; so the same comment holds. Thus the diagram will remain valid.

First, we promised that $stp(a_1a_2/e)$ can be chosen regular, $\doteq p$. Let $E = \{e' \in dcl(a_1a_2e) : w_p(e'/e) = 0\}$, $B = \{e' \in E : w_p(e'/\emptyset) = 0\}$ Since $stp(B/\emptyset)$ is hereditarily orthogonal to p, each of a_1, a_2, e realizes p over it; so we may absorb it into the base, i.e. we may assume $acl(B) \subset dcl(\emptyset)$. Note that $e \in E$, and $w_p(E/e) = 0$. Thus E/B is p-simple, of p-weight 1. By the regularity criterion above and the definition of B, $stp(E/B)$ is regular. Thus we may replace e by E. Since $stp(a_1a_2/E)$ is clearly p-simple of weight 1, another use of the regularity criterion shows that $stp(a_1a_2/e)$ is indeed regular.

Now let us change d. Choose $d' \in Cb(stp(d/\{a_1, a_2, b_1, b_2\})$ such that $d' \not\perp \{a_1, b_2\}$ and $d' \not\perp \{a_2, b_1\}$. d' has the obvious advantage of being algebraic over $\{a_1, a_2, b_1, b_2\}$. Therefore $stp(d'/\emptyset)$ is automatically p-simple. To compute its p-weight, recall that d' is definable over some Morley sequence $d^1,\dots,d^n$ in $stp(d/\{a_1,a_2,b_1,b_2\})$. Since d^1,d^2 are conjugate to d over $\{a_1,a_2,b_1,b_2\}$, each forks with $\{a_1,b_2\}$ as well as with $\{a_2,b_1\}$; so $w_p(a_i,b_{3-i},d^1,d^2)=2$. It follows that $w_p(d^2/d^1)=0$: otherwise $w_p(d^1,d^2)=2$, so by additivity $w_p(a_i,b_{3-i}/d^1,d^2)=0$ for $i=1,2$, giving $w_p(a_1b_1a_2b_2d^1d^2)=0+0+2=2\geq w_p(a_1b_1a_2)=3$. Similarly $w_p(d^i/d^1)=0$ for each i. Thus $w_p(d'/d^1)\leq w_p(d^1,d^2,\dots,d^n/d^1)=0$, so $w_p(d'/\emptyset)=1$. (It is not 0 since $\{a_1,b_2\}\not\perp d'$). Let $d''=\{\sigma d' : \sigma\in Aut(\mathbb{C}/\{a_1,a_2,b_1,b_2\})$. ($d'$ has only finitely many conjugates over this set.) So $d'\in acl(d'')$. Exactly the same argument as before shows that $d''/\emptyset$ is p-simple, of weight 1; and now we have $d''\in dcl(\{a_1,a_2,b_1,b_2\})$. If $d''/\emptyset$ is not regular, absorb $B''=\{b\in dcl(d''): w(b/\emptyset)=0\}$ into the base as before, so that $stp(d''$/the new base) is regular and no harm has been done. Note that $d\not\perp d'$, so $d\bowtie d'$. Replace d by d'. We now have $d\in dcl(\{a_1,a_2,b_1,b_2\})$.

Finally, repeat the previous paragraph with d_1,a_1,b_1,d,e in place of d,a_2,b_1,a_1,b_2 (in this order). The result is to have $d,d_1\in dcl(e,a_1,a_2,b_1,b_2)$. Let $\overline{d}=d_1d$, $\overline{a}=a_1a_2$, $\overline{b}=b_1b_2$. By the properties of our diagram, $\overline{d}\perp a_1\,|\,e$. Since $\overline{a}/e$ is regular, $\overline{a}\trianglelefteq a_1/e$, so $\overline{a}\perp\overline{d}\,|\,e$. Since $\overline{d}\in dcl(e,\overline{a,b})$ and $stp(\overline{b}/e,\overline{a})$ is regular, $stp(\overline{d}/e)$ is regular. So each of $\overline{d},\overline{a}$, and $\overline{b}$ realize a regular type over e, and they are pairwise independent over e. (Each is dominated over e by its first co-ordinate.) Write $\overline{d}=f(\overline{a,b})$ with f an e-definable function, and let $\overline{p}=stp(\overline{a}/e)$.

<u>Claim</u> If $\overline{a}'\models\overline{p}$, $\overline{a}'\perp\overline{b}\,|\,e$, and $f(\overline{a}',\overline{b})=\overline{d}$, then $\overline{aa'}\perp\overline{b}\,|\,e$.

<u>Proof</u> Since $tp(\overline{abe})=tp(\overline{a'be})$, and $\overline{d}$ is the same definable function of both sides, $tp(\overline{abde})=tp(\overline{a'bde})$. By the diagram, $a_1\not\perp\{d_1,b_1\}$ and $a_1\not\perp\{d,b_2\}$; so also $a_1'\not\perp\{d_1,b_1\}$ and $a_1'\not\perp\{d,b_2\}$. This pins down the location of a_1 and a_1' too closely for them to be independent; the argument is the same as one given earlier. [$w_p(d_1,b_1,a_1,a_1')=w_p(d,b_2,a_1,a_1')=2$ but it is not the case that

the p-weight of everything together is 2.] Thus $a_1 \perp a_1'$, so by transitivity $a_1 \perp a_1' | e$. So $\overline{a} \perp \overline{a'} | e$. By regularity of $\overline{a}/e, \overline{a'}/e$ and $\overline{b}/e$, it follows that $\overline{aa'} \perp \overline{b} | e$.

Define equivalence relations on realizations of $\overline{p}$: $x^1 E_L x^2$ if for $y \models \overline{p} | \{e, x^1, x^2\}$, $f(x^1, y) = f(x^2, y)$; and dually $y^1 E_R y^2$ if for $x \models \overline{p} | \{e, \overline{y}^1, y^2\}$, $f(x, y^1) = f(x, y^2)$. Let $a = \overline{a}/E_L$, $b = \overline{b}/E_R$, $p_L = stp(a/e)$, $p_R = stp(b/e)$. The above claim (and its dual) can now be restated as follows: If $\overline{a}, \overline{b} \models \overline{p} \otimes \overline{p}$, $a = \overline{a}/E_L$, $b = \overline{b}/E_R$ then the value of $f(\overline{a,b})$ depends on a and b; call it $f(a,b)$. Moreover, if $f(a,b) = f(a',b)$ then $a = a'$, and dually. (Writing $f(a,b)$ implies $a, b \models p_L \otimes p_R$.) This gives a weight 1 family of invertible germs on a regular type $\sqsubseteq p$, and so finishes the first part of the proof.

Lemma 1 Let q be p-simple, p locally modular, $q \sqsubseteq p^{(n)}$. Then every family of germs of permutations of q has p-weight $\leq n$.

Proof Let σ be a germ of a definable permutation of p; we will show $w_p(\sigma) \leq n$. σ is p-simple by 3.1.3. Let $a \models q | \sigma$. By modularity there exists c such stp(c) is p-simple, $w_p(c/\sigma) = w_p(c/a, \sigma a) = 0$ and $w_p(\sigma, a, \sigma a/c) = w_p(\sigma/c) + w_p(a, \sigma a/c)$. Since $stp(a/\sigma) \sqsubseteq$(a power of p), $a \perp c | \sigma$ so $a \perp \{\sigma, c\}$. Thus $w_p(\sigma/c) = w_p(\sigma/a, c)$, so $w_p(\sigma, a, \sigma a/c) = w_p(\sigma, a/c) = w_p(\sigma/c) + w_p(a/\sigma, c) = w_p(\sigma/c) + w_p(a/c)$. Subtacting this from the first equality, we get $w_p(\sigma a/a, c) = 0$. Let $a, a_1, a_2, \ldots$ be a long Morley sequence over $\{\sigma, c\}$. So $w_p(\sigma a_i/a_i, c) = 0$, and by additivity $w_p(\sigma a_1, \sigma a_2, \ldots/\{c, a_1, a_2, \ldots\}) = 0$. Since $a \perp \{\sigma, c\}$, $\langle a_1, a_2, \ldots \rangle$ is a Morley sequence also over $\emptyset$. By the proof of 3.1.3, $\sigma \in dcl(a_1, a_2, \ldots, \sigma a_1, \sigma a_2, \ldots)$. Thus $w_p(\sigma/c) = w_p(\sigma/c, a_1, a_2, \ldots) \leq w_p(\sigma a_1, \sigma a_2, \ldots/c, a_1, a_2, \ldots) = 0$. So $w_p(\sigma) \leq w_p(c) = w_p(c/a) \leq w_p(c/\{\sigma a, a\}) + w_p(\sigma a/a) = 0 + n$, as the lemma states.

Corollary 2 Let p be a locally modular regular type (stationary over the base set), and let σ be an invertible germ of a definable function on p into r, with $q=stp(\sigma)$ regular, $\not\perp p$. Let $q^\circ=stp(\tau^{-1}\cdot\rho)$ for $\tau,\rho\models q\otimes q$. Then q° is regular, $\not\perp p$, and is closed under generic composition: if $\sigma_1,\sigma_2\models q^\circ\otimes q^\circ$ then $\sigma_1\cdot\sigma_2\models q^\circ$.

Proof q° is clearly p-simple. Let $\sigma\models q^\circ$, so $\sigma=\tau^{-1}\cdot\rho$ with $\tau,\rho\models q\otimes q$. Then $w_p(\sigma/\tau)=w_p(\rho/\tau)=1\leq w_p(\sigma/\emptyset)$, so by (4) we have equality, and $\sigma\perp\tau$. Similarly $\sigma\perp\rho$. Since $\sigma\in dcl(\rho,\tau)$ and τ/ρ is regular, q° is regular. Let $g\sigma_1,\sigma_2\models q^\circ\otimes q^\circ$. Let $\rho\models q|\sigma_1,\sigma_2$. So $\sigma_i=\rho^{-1}\cdot\tau_i$ for $i=1,2$. Thus $\sigma_1{}^{-1}\cdot\sigma_2=(\rho^{-1}\cdot\tau_1)^{-1}\cdot\rho^{-1}\cdot\tau_2=\tau_1{}^{-1}\cdot\tau_2$. Now τ_1,τ_2 are independent over ρ since σ_1,σ_2,ρ are independent, and $\tau_1\perp\rho$; so $\tau_1\tau_2\models q\otimes q$, and $\sigma_1{}^{-1}\cdot\sigma_2\models q^\circ$, showing that q° is closed under generic composition.

Combining the corollary with the existence of a nontrivial germ proved above, and with theorem 3.1.1, we get an Abelian group A with regular generic type $\not\perp p$. This finishes part (a).

It remains to fill in the details with respect to the division ring. First the fact that every definable endomorphism is small. Let $\sigma:A\to A/S$ be a definable homomorphism of A, with $S\subset A_0$. Let b be a generic element of A/S. If $\sigma x+b=\sigma' x+b'$ generically, then $\sigma-\sigma'$ is generically constant, hence 0, so $\sigma=\sigma'$ and $b=b'$. Thus both σ and b are definable from the germ of the map $x\mapsto\sigma x+b$. By the lemma, $w_p(\{\sigma,b\})\leq 1$. So $w_p(\sigma)=0$, i.e. σ is small.

We have to show that $a_1,\ldots,a_n\in A$ are independent iff their images modulo A_0 are D-independent. One direction is obvious: D consists of small endomorphisms; so if $\overline{b}=\Sigma\alpha_i\overline{a_i}$ with $\alpha_i\in D$, $\overline{a_i}=a_i+A_0$, then

$w_p(b/a_1,a_2,\ldots)\le w_p(b/a_1,a_2,\ldots,\alpha_1,\alpha_2,\ldots)+w_p(\alpha_1,\alpha_2,\ldots)=0+0$. The other direction is equivalent to the following lemma.

Lemma 3 Let G be a p-simple group, of p-weight m, and let q be a p-simple strong type of elements of G, with $q\Box p^n$. Assume p is locally modular. Then q is a translate of the generic type of some subgroup of G.

Proof We will treat q and its translates as global types (over $\mathbb{C}$), and assume q is based on $\emptyset$. G acts on the set of global 1-types of G by: $(d,r)\mapsto {}^d r$, where ${}^d r=stp(d\cdot x/\mathbb{C})$ for $x\models r|\mathbb{C}$. Let $d\in G$ be generic, and let $a\models q|d$. Then $stp(d\cdot a/d)\Box stp(a/d)=q\Box p^n$, and $stp(d/a\cdot d)\Box stp(a/a\cdot d)=(q|a\cdot d)\Box p^n$. (Using the genericity of d.) By proposition 5.1.5, $w_p({}^d q)+w_p(Cb({}^d q))=w_p(d\cdot a)=m$. Thus $w_p(Cb({}^d q))=n-m$.

Let $S=Stabilizer(q)=\{d\in G: {}^d q=q\}$, and let $\overline{d}$ denote the equivalence class of d under the definable equivalence relation relation: $xS=yS$. Note that $Cb({}^d q)=dcl(\overline{d})$: it suffices to see that an automorphism σ of $\mathbb{C}$ fixes (the parallelism class of) ${}^d q$ iff it fixes dS; and indeed $\sigma({}^d q)={}^{\sigma(d)}q$, and ${}^d q={}^{d'}q$ iff ${}^{d^{-1}\cdot d'}q=q$ iff $d^{-1}\cdot d'\in S$. Thus $w_p(\overline{d})=m-n$. Since $w_p(d)=m$, $w_p(S)=w_p(stp(d/\overline{d}))=m-(m-n)=n$.

Let $\tau_0\models q$. Let $\sigma\in S$ be generic over τ_0. Then by definition $\sigma\cdot\tau_0\models q|\emptyset$. Moreover $w_p(\sigma\tau_0/\tau_0)=w_p(\sigma/\tau_0)=n=w_p(q)$, so by the second property of local weight, $\sigma\tau_0\perp\tau_0$, i.e. $\sigma\tau_0\models q|\tau_0$. It follows that every realization of q is in the same right coset of S, so this right coset is 0-definable, i.e. $C=\emptyset$, and q is a (right) translate of the generic type of S.

Now we can prove part (b) of the theorem. Let $a_1,\ldots,a_n,b\in A-A_0$. Suppose $b \not\perp a_1\ldots a_n$. Without loss of generality, $a_1,\ldots,a_n$ are independent.

Augment the base set by $\{c \in acl(a_1,\ldots,a_n,b): w_p(c/\emptyset)=0\}$ Note that this does not change the definition of A_0, etc. By the regularity criterion, $q=stp(a_1,\ldots,a_n,b)$ is $\Box p^n$. By the lemma, there exists a subgroup S of $A^n \times A$ such that q is a translate of the generic type of S. Let $D=\{(x_1,\ldots,x_n): (\exists y)(x_1,\ldots,x_n,y) \in B\}$, and let $S=\{y:(0,\ldots,0,y) \in B\}$. It is easy to see that D contains every generic element, hence $D=A^n$; and that S does not contain a generic, so $S \subset A_0$. So B is the graph of a homomorphism $\sigma: A^n \to A/S$. (To be precise, find a 0-definable group S' such that $A \cap S'$ is a proper subgroup of A containing S, and replace S by S' and σ by the induced homomorphism.) Let $\sigma_i(x)=\sigma((0,\ldots0,x,0,\ldots0))$ with x in the i'th place. Then $\sigma_i: A \to A/S$ is a homomorphism, and $\sigma(\overline{x})=\Sigma\sigma_i(x_i)$. Now the fact that q is a translate of some type inside B implies that some translate of $\{(x_1,\ldots,x_n,y): y=\Sigma\sigma_i(x_i)\}$ is in q. Since $dom(\sigma)=A^n$, this translates into saying that $q \vdash y=\Sigma\sigma_i x_i+c$ for some c. Since this clearly determines c as a function of q and the σ's, and they are all defined over $acl(\emptyset)$, $c \in acl(\emptyset) \subset A_0$. So b is D-linearly dependent on the a_i's.

Call a group p-complemented if it is definably a factor of a finite direct sum of connected groups with regular generic type $\Box p$.

Corollary 4 Let G be connected and p-simple, p locally modular. Then G has a normal subgroup N such that N is hereditarily orthogonal to p, and G/N is p-complemented. If G is p-semi-regular, then G is Abelian.

Proof There exists a stationary regular type $q \Box p$ of elements of G. Therefore by the lemma, there exists a connected subgroup A of G with regular generic type, $\Box p$. Let n be the p-weight of the generic type of G. So over a possibly larger base set, there exist a generic $b \in G$ and

$a_1,a_2,\ldots,a_n \models q^n$ such that $b \not\perp a_i$ for each i. Using the lemma again, $stp(ba_1a_2\ldots a_n/\{e \in acl(ba_1\ldots a_n): w_p(e)=0\})$ is a translate of a subgroup of $G \times A \times \ldots \times A$; it is easy to see that we can actually take it to be a subgroup. In other words, it defines a homomorphism $h: G \to A'$, where A' is a factor of the A^n. (The domain of the homomorphism is G since it contains the generics.) So N=Ker(h) works. If the generic of G is $\sqsubset p^n$, then $stp(\sigma\tau\sigma^{-1}\tau^{-1}/\emptyset) \trianglelefteq p^{2n}$ when σ,τ are independent generic elements; but G/N is Abelian, so $[\sigma,\tau] \in N$, and hence $w_p(\sigma\tau\sigma^{-1}\tau^{-1}/\emptyset)=0$. This forces $\sigma\tau\sigma^{-1}\tau^{-1} \in acl(\emptyset)$, and by connectedness $\sigma\tau\sigma^{-1}\tau^{-1}=1$, i.e. G is Abelian.

In particular in a locally modular superstable theory, every group is solvable. This should be compared with Pillay's result that simple superstable groups are nilpotent [P2].

The lemma can also be used to show that D is independent of the base, giving an honest proof that every endomorphism is small. To do this let us describe D in another way. Recall the following facts from §3.3.

Fact 5 Let G be p-simple. Then G has a unique p-semi-regular subgroup N such that the generics of G/N are orthogonal to p. N is normal.

Fact 6 Let G be p-semi-regular with generic q, and let r be a strong type of elements of G. If $w_p(q)=w_p(r)$, then q=r.

Notation If A,B,C are Abelian groups, $E \subset A \times B$ and $F \subset B \times C$ subgroups, let $F \cdot E = \{(x,y) \in A \times C: \exists z \in B.\ (x,z) \in E\ \&\ (z,y) \in C\}$, and $E^{-1} = \{(x,y): (y,x) \in B\}$.

Note that the generic types of $E \cdot F$ and of E^{-1} have p-weight 1 provided that A,B,C,E,F are regular, $\sqsubset p$.

Now if A is regular, the division ring D(A) can be described as the set of regular subgroups E of $A \times A$ other than $(0) \times A$; $\alpha \in D$ corresponds to the unique regular subgroup of $\{(x,y): y = \alpha x \ (\mathrm{mod}\ A_0)\}$ given by fact 6. Multiplication being given by the operation $E,F \to (E \cdot F)^\circ$, where $(E \cdot F)^\circ$ denotes the unique regular subgroup of $E \cdot F$. So the independence from the base comes from:

Corollary 7 If A is p-simple, $B \subset A$ p-semi-regular, and p is locally modular, then B is definable with orthogonal parameters; i.e. there exists X such that B is ($\bigwedge$)- definable with parameters from X and $\mathrm{stp}(X/\varnothing)$ has p-weight 0.

Proof By 3.1.2, B is the intersection of definable groups B_i. The maximal p-semi-regular subgroup N_i of B_i given by Fact 6 must contain B, so B is the intersection of the N_i's. If B_i is definable from $\bar{a}$, then by the uniqueness of N_i it must also be ($\bigwedge$-)definable over $\bar{a}$. So it suffices to consider the case in which B is ($\bigwedge$-)definable over a finite set $\bar{a}$.

Let b realize the generic type of B over $\bar{a}$. Let $X = \mathrm{acl}(\overline{ab}) \cap \mathbf{Cl_p}(\varnothing)$. Consider $r = \mathrm{stp}(\overline{ab}/X)$. (Consider $\overline{ab}$ as an element of some Cartesian power of A.) By the lemma, there exists a subgroup S such that r is the generic type of a coset of S. In particular, if b' is an element of B generic to $\bar{a}$,b, then $\overline{ab}' \models r$, so $(\bar{0},(b-b')) \in S$. It follows that $B \subset \{x: (\bar{0},x) \in S\}$. Conversely, if $(\bar{0},c) \in S$ and $c \perp \bar{a},b$ then $(\bar{a},b+c)$ is in the same coset as $(\bar{a},b)$ and is still generic, so $b+c \in B$, hence $c \in B$. Thus $B=S$ is X-definable.

(By the way, despite the way the corollary is stated, it is obvious that groups do have a canonical base of definition.)

During the proof parameters were used freely. The situation in this respect is as follows. Example 1 showed that orthogonal parameters need to be added in order to find a regular representative of ▭p. Then one may need to add one parameter ▭p. Above the algebraic closure of this, the group may be found. (The proof calls for augmeting the base set first to make p modular and non-trivial over it, but this can easily be shown to be unneccessary using proposition 5.1.1). The neccessity of one p-parameter is shown by the simplest example, infinite dimesional projective space over a field (finite or algebraically closed). However, the following proof shows that this is essentially the only way of postponing the existence of the group.

Theorem 5.3 Let p be locally modular. If p is non-orthogonal to B, then there exists a regular type q based on $\mathbf{Cl}_p(B)$ such that q▭p and the geometry on q is modular.

Proof

<u>Case 0</u> p is trivial. Then every q▭p is trivially modular.

Assume p is non-trivial.

CONVENTIONS Fix the regular type p. For the rest of this proof "regular" will mean "regular and ▭p". Given a p-simple group A, denote the maximal p-semi-regular subgroup of Fact 5 by A°. (This does not agree with standard notation.)

By Theorem 3, p may be assumed to be based on $\mathbf{Cl}_p(B)$. By theorem 2, there exists a regular group A. We noted above that A=A(e) is e-definable for some e with $w_p(e/\varnothing)=1$. By working over $acl(e)\cap Cl_p(\varnothing)$, we may assume $stp(e/\varnothing)$ is regular; call it p again. The letters e and d and their variants will be reserved for realizations of p, and the base set ($\subset \mathbf{Cl}_p(B)$) will be suppressed.

Let e,e' be independent. Then the generics of A(e),A(e') are modular and non-orthogonal, so there exist generic $a \in A(e)$ and $a' \in A(e')$ such that $a \not\perp a' \mid \{e,e'\}$. Let $s = stp(aa'/Cl_p(\{e,e'\}))$. Then s is regular, and by the last lemma it is a translate of the generic type of some regular group $F_0(e,e') \subset A(e) \times A(e')$. The $F_0(e,e')$'s form a system of "almost-isomorphisms" between the A(e')s; we shall modify it so that the system almost commutes. Given e_0,e_1 and $e \perp \{e_0,e_1\}$, let $F(e_0,e_1,e) = [F_0(e_1,e)^{-1} \cdot F_0(e_0,e)]^\circ$. So $F(e_0e_1,e)$ is a regular subgroup of $A(e_0) \times A(e_1)$. By lemma 7, $F(e_0,e_1,e)$ may be defined with parameters from X for some X such that $w_p(X/\{e_0,e_1\})=0$. In particular, $X \perp e \mid \{e_0,e_1\}$; so $F(e_0,e_1,e)$ does not in fact depend on e, and may be written as $F(e_0,e_1)$. This gives a measure of commutativity:

(*) For any e_0,e_1, $F(e_0,e_1)$ is a regular subgroup of $A(e_0) \times A(e_1)$. Given e_0,e_1,e_2, $F(e_0,e_2) = [F(e_1,e_2) \cdot F(e_0,e_1)]^\circ$. F(e,e) is the diagonal subgroup. $F(e,e') = F(e',e)^{-1}$.

Let $A_0(e) = \{x \in A(e) : w_p(x/e) = 0\}$, $K(e,e') = \{x \in A(e) : (x,0) \in F(e,e')\}$.

<u>Case 1</u> $K(e,e') \subset A_0(e)$ for $e \perp e'$.

In this case we have a commuting system of isomorphisms betwen the groups $A(e)/A_0(e)$, and only some easy definablity considerations need to be met before concluding that the e's were never really neccessary to define the group. Call a subset of A(e) relatively definable if it has the form $J \cap A(e)$, where J is a definable set; we would like to replace $A_0(e)$ by a relatively definable subgroup, and mod out by it. By theorem 1, $F(e,e')$ is the intersection of $\{e,e'\}$-definable groups $F_i(e,e')$. Since $F(e,e')=F(e',e)^{-1}$, the F_i's can be chosen so that $F_i(e,e')=F_i(e',e)^{-1}$. Let $K_i(e,e')=\{x: (x,0)\in F_i(e,e')\}$. So $\cap_i K_i(e,e')=K(e,e')\subset A_0(e)$. By compactness, $K_i(e,e')\subset A_0(e)$ for some i. ($A_0(e)$ being the complement of a $\bigwedge$-definable set.) By stability, there exists $X\subset A_0(e)$ such that $K_i(e,e')$ is X-definable. Thus $K_i(e,e')$ is both X-definable and $\{e,e'\}$-definable; since $X\perp e'|e$, it must be acl(e)-definable. [$X\perp e'|e$ because $stp(X/e)\perp p$]. So $K_i(e,e')$ does not truly depend on e'. Call it K(e), and let $\hat{A}(e)=A(e)/K(e)$, and $\hat{F}_0(e,e')=\{(x+K(e),y+K(e')): (x,y)\in F_i(e,e')\}$. So for $e\perp e'$, $\hat{F}_0$ is the graph of an isomorphism $\hat{A}(e)\rightarrow\hat{A}(e')$. Noting that $\hat{A}(e)$ is still a regular $\bigwedge$-definable group, define $\hat{F}(e,e')$ for arbitrary e,e' from $\hat{F}_0$ just as F was defined from F_0. Then we have a commuting system as in (*), only now the $\hat{F}(e,e')$'s are actual isomorphisms. The direct limit B of this system is $\bigwedge$-definable without parameters, and is e-isomorphic to any $\hat{A}(e)$.

<u>Case 2</u> $K(e,e')\not\subset A_0(e)$ for $e\perp e'$.

Recall the discussion of composition of subgroups on page 89. The division ring D(A) associated with a regular group A was identified there with the set of regular subgroups of A×A other than (0)×A. If A,A' are two regular groups, and $I\subset A\times A'$ is a regular subgroup other than (0)×A' and A×(0), then we get an isomorphism $\hat{\imath}:D(A)\rightarrow D(A')$ given by $E\rightarrow(I^{-1}\cdot E\cdot I)^{\circ}$. In particular, F(e,e') gives rise to an isomorphism $D(A(e))\rightarrow D(A(e'))$. So we have a commuting system of isomorphisms between the D(A(e))'s. Thus we may as well identify them all with one abstract division ring D. By the definition of this identification,

(**) If $(x,y) \in F(e,e')$ and $\alpha \in D$ then there exist $x' \in A(e)$, $y' \in A(e')$ such that $x' = \alpha x \pmod{A_0(e)}$, $y' = \alpha y \pmod{A_0(e')}$ and $(x',y') \in F(e,e')$.
(The set of pairs (x,y) satisfying the statement is a subgroup of $F(e,e')$ which is not p-trivial, hence equals $F(e,e')$.)

Claims

1) $F(e'',e') \cdot F(e,e'') = F(e,e') + (0) \times K(e',e'')$

2) If $e \perp e'$, $(x,y) \in F(e,e')$, and $x \notin A_0(e)$ then there exists $e'' \perp e,e'$ with $x \in K(e,e'')$ and $y \in K(e',e'')$.

3) Given $e \perp d$, $z \in K(e,d)$, and $\alpha \in D$, there exist $\hat{z}$ and $\hat{d}$ such that $\hat{z} = \alpha z \pmod{A_0(e))}$, $\hat{z} \in K(e,d')$, and $d \not\perp \hat{d} \mid \emptyset$.

4) Given $e \perp \{d_1,d_2\}$, and given $z_i \in K(e,d_i)$ $(i=1,2)$ and $\alpha,\beta \in D-(0)$ there exist $d \in \overline{\mathbf{Cl}}_p(d_1,d_2)$ and $y \in K(e,d)$, such that $y = \alpha z_1 + \beta z_2 \pmod{A_0(e)}$.

Proofs

1) $\supset$ is clear. Conversely let $(x,y) \in$ LHS. So $(x,z) \in F(e,e'')$, $(y,z) \in F(e',e'')$. By (*) for ee'', there exists y' s.t. $(x,y') \in F(e,e')$ and $(z,y') \in F(e'',e')$, equivalently $(y',z) \in F(e',e'')$. So $(y-y',z-z) \in F(e',e'')$, i.e. $y-y' \in K(e',e'')$.
$(x,y) = (x,y') + (0,y-y')$

(2) Let e,e',e''' be independent. By (*) for e,e''', if $(x_0,0) \in F(e,e''')$ then there exists $y_0 \in A(e')$ such that $(x_0,y_0) \in F(e,e')$ and $(0,y) \in F(e''',e')$. Thus $(x_0,y_0) \in F(e,e') - A_0(e) \times A(e')$ and (x_0,y_0) satisfy the required conclusion. By Fact 5, $F(e,e') - A_0(e) \times A(e')$ is the extension of a complete type over $\{e,e'\}$, namely the generic type of $F(e,e')$. Hence the conclusion holds also for (x,y).

3) Let $e' \perp e,d$, $z' \in K(e',d) - A_0(e')$. By (1), z' may be chosen so that $(z,z') \in F(e,e')$. By (**), there exist $\hat{z} \in A(e)$, $\hat{z}' \in A(e')$ such that $\hat{z} = \alpha z$ (mod

$A_0(e))$, $\hat{z}'=\alpha z'$ (mod $A_0(e')$) and $(\hat{z},\hat{z}')\in F(e,e')$. By (2), there exists $\hat{d}$ such that $\hat{z}\in K(e,\hat{d})$ and $\hat{z}'\in K(e',\hat{d})$. Since $\hat{z}\in \mathbf{Cl}_p(z,e)\subset \mathbf{Cl}_p(d,e)$, $\hat{d}\not\perp\{d,e\}$. Similarly $\hat{d}\not\perp\{d,e'\}$. This forces $\hat{d}\not\perp d$.

4) This is the sum of three special cases: $\alpha=0$, $\beta=0$, and $\alpha=\beta=1$. The first two follow from (3). The third has the same proof, but with (**) replaced by the fact that $F(e,e')$ is closed under $+$.

CONCLUSION: In case 3, p is modular.

Proof: Suppose not. It is easy to see, using 5.1.1 and 5.1.3, that the only possible obstruction to modularity is the existence of parallel lines, i.e. of elements d_1,d_2,d^1,d^2 such that $d_1d_2\not\perp d^1d^2$ but there is no realization of p in $\mathbf{Cl}_p(d_1d_2)\cap\mathbf{Cl}_p(d^1d^2)$. Pick $e\perp d_1d_2d^1d^2$, and pick $z_i\in K(e,d_i)-A_0(e)$, $z^i\in K(e,d^i)-A_0(e)$ $(i=1,2)$. Clearly $z_1z_2\not\perp z^1z^2|e$, so $\alpha_1z_1+\alpha_2z_2=\alpha^1z^1+\alpha^2z^2$ (mod $A_0(e)$) for some $\alpha^1,\alpha^2,\alpha_1,\alpha_2\in D$. If $\alpha^1\alpha^2\alpha_1\alpha_2=0$ then the situation is trivial, so assume otherwise. By claim 4, there exist $d\in\mathbf{Cl}_p(d_1d_2)$ and $z\in K(e,d)$, with $z=\alpha_1z_1+\alpha_2z_2$, and by another application there exist $d'\in\mathbf{Cl}_p(d^1d^2)$ and $z'\in K(e,d')$ with $z'=\alpha^1z^1+\alpha^2z^2$ (mod $A_0(e)$). So $z=z'$ (mod $A_0(e)$), and hence $d\not\perp d'|e$. Since $d,d'\in\mathbf{Cl}_p(d_1d_2d^1d^2)$, we have $\{d,d'\}\perp e$, so $d\not\perp d'|\varnothing$. Thus $d\in Cl(d_1d_2)\cap Cl(d^1d^2)$. This shows that there does exists a realization of p in $\mathbf{Cl}_p(d_1d_2)\cap\mathbf{Cl}_p(d^1d^2)$. So there are no parallel lines, and the geometry is projective over D.

Corollaries Let p be regular, locally modular, based on $\varnothing$.

7. If p is not modular, then there exists a regular group A, $\mathbf{A}$-definable over $acl(\varnothing)$, a strong type $\hat{p}^a\sqsupset p$ based on $\varnothing$, and a definable action of A on the extension of $\hat{p}$. A acts regularly on $\hat{p}$, and as a group of automorphisms.

8 The geometry on p is trivial, or is isomorphic to the affine or projective space over a division ring.

9. If $p^a \perp q$ then $p^a \perp q^\omega$ or $q^a \perp p^\omega$. (q regular.)

These corollaries were found in discussions with Chris Laskowski. Originally we assumed the existence of a 0-definable group; theorem 3 gives one for free.

Proofs

7. Let r be the modular representative given by the theorem. If $q^a \not\perp r^\omega$ over $\emptyset$, then by [CHL] or [Bu3] $q^a \not\perp r$, so q must be modular. If $q^a \perp r^\omega$ the assertion follows from Theorem 4.1.

8. If the geometry on p is modular then this is the "fundamental theorem of projective geometry"; it is also implicit in the proof of theorem 3. If it is not modular, then the conclusion of (7) holds: $\hat{p}$ is an affinized version of A. For $x_0, x_1 \models \hat{p}$, let $x_1 - x_0$ be the unique element of A that sends x_0 to x_1. Let $x_0, \ldots, x_n \models \hat{p}$. Then $(x_0, \ldots, x_n)$ and $(x_1 - x_0, \ldots, x_n - x_0)$ are bi-definable over x_0. Since $\hat{p}^a \perp r^\omega$, $w_p(x_0 \ldots x_n) = n+1$ iff $w_p(x_1 - x_0, \ldots, x_n - x_0) = n$. In particular, $x_0 \not\perp x_1$ iff $x_1 - x_0 \in A_0 = \{a \in A: w_p(a) = 0\}$. Thus we have an induced regular action of $V = A/A_0$ on $G = \hat{p}/\not\perp = \{x/\not\perp : x \models \hat{p}\}$. By 2(b), V is a vector space over a division ring D, and forking dependence = linear dependence. The action of V on G induces a D-affine structure on G, and the statement about p-weights means that forking dependence = affine dependence on G.

9. This was proved in [CHL] from two axioms: local modularity and "uniqueness of parallel lines." (8) says precisely that the second axiom is redundant.

Here is one last very special case, in which the group may be taken to be definable with orthogonal parameters. Most of its power was lost once theorem 3 was proved (making the proposition obvious for countable theories, for example.) It will nevertheless be included, as it illustrates an entirely different way of finding a group translation. (And it was promised in §3.)

Proposition 10 Let p be a weakly minimal, non-trivial type over an algebraically closed set B. Then either there exists a B-definable Abelian group A, with $R^{\infty}(A)=1$ and p $\not\perp$ each generic type of A, or else p is strongly minimal.

Proof

Absorb B into the signature. Assume p is not strongly minimal. By [Bu3], p is locally modular. It follows that the non-triviality of p can be demonstrated without extra parameters: for $a_1,a_2 \models p^{(2)}$ there exists an element $b \in \mathbb{C}^{eq}$ realizing a regular type, such that the algebraic closure of any two of a_1,a_2,b contains the third. Choose b so as to minimize $\mathrm{Mult}(a_1/a_2b)$. Let $q=\mathrm{stp}(b/\emptyset)$. Note that $R^{\infty}(q)=1$. Let $\theta(x_1,x_2,y)$ be a formula true of a_1,a_2,b such that θ generates the type of each of them over the other two, and implies a formula of $R^{\infty} \doteq 1$ about each individually. Define an equivalence relation on $q^{\mathbb{C}}$ by: $y \sim y'$ if for generic $x_2 \models p$, $\models (\forall x_1)(\theta(x_1,x_2,y) \equiv \theta(x_1,x_2,y'))$. Clearly we may replace q by $\mathrm{stp}(b/\sim)$; so we may assume $\sim$ is the identity. Now let $b \models q$, and let b' be arbitrary. Let $a \models p|\{b,b'\}$. Suppose there exists c s.t. $\theta(c,a,b)$ & $\theta(c,a,b')$. Since $\models \theta(c,a,b)$, and $ab \models p \otimes q$, it follows that $ca \models p \otimes p$; so by the second conjunct, $b' \models q$. Since $b,b' \in \mathrm{acl}(c,a)$, $U(c,a,b,b') \leq 2 = U(a,b)$. Thus $b' \in \mathrm{acl}(b)$. Now if

$b \neq b'$, then $\{x: \theta(x,a,b)\} \neq \{x: \theta(x,a,b')\}$, so the intersection of these two sets is smaller than either, and thus $Mult(c/a,b,b') < Mult(c/a,b)$. This contradicts the minimality in the choice of b. Thus $b=b'$. We have shown that if $b \models q$ and if for $a \models p|\{b,b'\}$, $(\exists x)(\theta(x,a,b) \& \theta(x,a,b'))$, then $b=b'$.

It follows that if $a_1a_2 \models p^2$, then $\theta(a_1,a_2,b)$ & $\theta(a_1,a_2,b') \Rightarrow bb' \models q^{(2)}$ or $b=b'$. Thus $a_1a_2 \in acl(bb')$ if $b \neq b'$; so there exists a formula $\alpha(x,y_1y_2)$ such that $\models(\forall y_1y_2)(\exists \leq \ell$ elements $x)(\alpha(x,y_1,y_2))$ and $\models(\forall y_1y_2)(y_1 \neq y_2$ & $\theta(a_1,a_2,y_1)$ & $\theta(a_1,a_2,y_2) \Rightarrow \alpha(a_1,y_1,y_2))$. Denote this last true sentence by $\rho(a_1a_2)$. Let $\psi(x) = (\text{for } y \models q|x)(\exists x_2)(\theta(x,x_2,y) \& \rho(x,x_2))$. So $\models \psi(a)$ for $a \models p$.

Let $b^1b^2 \models q^{(2)}$. Then there are only ℓ elements x such that $\models \alpha(xb^1b^2)$. Let $p^1,\ldots,p^{\ell}$ be a list of all types realized by those elements. Let a_1 be such that $\models \psi(a_1)$, and $tp(a_1/B)$ is neither algebraic nor in the above list. (This is possible since p is not strongly minimal, so ψ does not have Morley rank 1.) Let $b \models q|a_1$. Pick a_2 s.t. $\theta(a_1a_2b)$ & $\rho(a_1,a_2)$. I claim that $b \in dcl(a_1,a_2)$. For suppose not. Let $b' \neq b$ be a conjugate of b over a_1a_2. Then $b' \models q$. Since $\models \rho(a_1,a_2)$ and $\theta(a_1a_2b)$ and $\theta(a_1a_2b')$, $\models \alpha(a_1,b,b')$. If $b \not\perp b'$ then $b' \in acl(b)$, so $b'b \perp a_1$; this, together with the fact that $a_1 \in acl(bb')$, implies $a_1 \in acl(\emptyset)$, a contradiction. If $b \perp b'$, on the other hand, then $bb' \models q^{(2)}$, and so $tp(a_2/B)$ is one of the p^i's, again a contradiction. So we have shown that $b \in dcl(a_1,a_2)$. Since $b \in acl(a_1,a_2)$ and $b \perp a_1$, $R^{\infty}(a_2/a_1) \geq 1$. Since $R^{\infty}(a_2) \leq 1$, it follows that $a_2 \perp a_1$ and $R^{\infty}(a_2)=1$. Similarly it is easy to see, since $R^{\infty}(a_1)=1$, that a_1,a_2,b are indpendent in pairs, and each realizes a type $\square p$. So we are back where we started, but now $b \in dcl(a_1,a_2)$.

Now we can repeat everything, but transposing the roles of a_1 and b. However, $Mult(b/a_2,a_1)=1$ is already as small as possible, so there is no need to replace a_1 (as b was replaced in the first paragraph.) b and a_2 will be changed as in the proof above; but as we are allowed to preserve the truth value of any given formula about the triple, we can ensure that $b \in dcl(a_2, a_1)$ stays true. The hypothesis that the original type p was not strongly minimal is also still valid for the new types. So we get a new triple a_1, a_2, b such that $a_1 \in dcl(a_2, b)$ *and* $b \in dcl(a_1, a_2)$. This means that a_2 defines a germ of an invertible function $tp(a_1) \leftrightarrow tp(b)$. By corollary 2 and theorem 3.1.1, we get a definable group.

§6. Finitely Based Theories

The following conjecture has been associated with Lachlan:

(*) If T is a countable, stable theory and the number of isomorphism classes of countable models is finite, then it equals 1.

Lachlan proved this for superstable theories. Using a different idea, Pillay proved it for 1-based theories. Buechler and Pillay have suggested a common generalization, that (*) is true for finitley based theories. This is proved below.

It is not completely clear how "finitely based" should be defined. Here are three possible definitions, together with the way they will actually be referred to:

1) Every nontrivial type is non-orthogonal to some finite set. ("T is finitely controlled".)

2) Every global type is based on some finite set. ("T admits finite coding")

3) Every set of indiscernibles is based on a finite subset. ("T is finitely based.")

It is clear that every 1-based or superstable theory satisfies (3), and (1)$\Leftarrow$(2)$\Leftarrow$(3). I think (3) is the most appropriate notion. The conjecture will in fact be proved for all theories satisfying (2).

To see how different (2) is from the usual "positive" properties, note that every stable theory is interpretible in one that has finite coding. Let M be any stable model, and S the interpretation of some sort of M. Let $P={}^{\omega}S=\{$all functions on ω with range $S\}$, and let π_n $(n<\omega)$ be the projections. Then $(M,P,\pi_1,\pi_2,\ldots)$ is stable. Iterating this process, one sees that every stable theory is interpretible in another with the additional property:

(*) for every sort S there exists a sort S* s.t. for every set $X \subset S^{\mathfrak{C}}$ with $|X|=|T|$ there exists an element $c \in S^*$ such that $X \subset dcl(c)$.

Clearly (2) is a consequence of (*). The theory of seperably closed fields K of characteristic p>0 satisfying $[K:K^p]=p$ admits finite coding for the same reason, ; it furnishes an example of (2)&-(3). It is not too difficult to modify it so as to get an example of a theory that has (1)&-(2).

We will also give solutions of Lachlan's conjecture on $\aleph_0$-categorical stable theories and of the strong Kueker conjecture for theories satisfying the weakest of our properties. These results were also suggested by Steve Buechler. The chapter originally closed with the easy proof of the following proposition, showing that only finitely controlled theories need be considered in work on "Morley's conjecture for $\aleph_1$-saturated models." The proof has been suppressed in view of my discovery of Shelah's Dimensional Discontinuity Property, which is a little weaker than the hypothesis of the proposition and implies the existence of many models. The fact does provide some additional incentive to study finitely controlled theories.

Fact Let T be countable and stable, and assume every model of T of cardinality κ extends to an $\aleph_1$-saturated one of the same cardinality. Let λ be a regular cardinal, $\kappa \geq \lambda > \omega$. Suppose there exists a type p based on some set B such that p is orthogonal to every finite subset of B. Then T has at least 2^λ non-isomorphic $\aleph_1$-saturated models of power κ.

Theorem 1 If T is a countable, stable theory and T admits finite coding then T is $\aleph_0$-categorical or T has infinitely many isomorphism classes of countable models.

proof Let us first show that T has a rudimentary theory of finite weight. More precisely, simply because T is small and admits finite coding, the following property holds:

(FW) Let B be a finite set, a a given element, and let I be an independent sequence of elements b satisfying: $b \not\perp a$ (over $\emptyset$). Then I is finite.

For suppose (FW) fails, and let a,B,I form a counterexample. WLOG $|I|=\aleph_0$. Say $I=\{c_\alpha : \alpha \in Q\}$, where Q is the set of rationals. Given a real number x, let $I_x=\{c_\alpha : \alpha<x\}$, and let $p_x=\mathrm{stp}(a/I_x)$. Since T admits finite coding, p_x is based on some finite set S_x. Let $a_x \models p_x | S_x$, enumerate S_x in some way, and let $q_x=\mathrm{tp}(S_x\hat{\ }a_x/\emptyset)$. I claim that it x,y are distinct reals then $p_x \neq p_y$. For say $x<y$. Then $I_x \subset I_y$. Since I is independent, $a \not\perp I_x | (B \cup I_y)$, so $rk_\Delta(p_x) > rk_\Delta(p_y)$ for some finite Δ. Thus $rk_\Delta(a_x/S_x) > rk_\Delta(a_y/S_y)$, so there is no automorphism carrying S_x to S_y and a_x to a_y, i.e. $q_x \neq q_y$. So assuming the negation of (FW) and finite coding, we have found continuum many types over $\emptyset$.

Thus FW does hold. The rest of the proof consists simply of noting that Lachlan's proof really used FW and not the full assumption of superstability. One shows, in fact, that FW implies the following property:

(P) Let $a \perp b$ and suppose $\mathrm{tp}(b/\emptyset)$ is not isolated. Then Pr(ab) cannot be embedded into Pr(a). (Pr(x) denotes the prime model over x).

For suppose otherwise. Then there exists a_0 such that $tp(ab/a_0)$ is isolated. In particular, a_0 weakly isolates ab, meaning that there exists a formula $\varphi(x,\overline{y})$ s.t. $\models\varphi(a_0,ab)$ and $\varphi(a_0,y)\models tp(ab/\emptyset)$. Define $a_1,a_2,\ldots, b_1,b_2,\ldots$ as follows: $a_1=a$, $b_1=b$. Given $a_0,\ldots,a_n$ and $b_1,\ldots,b_n$, pick a_{n+1},b_{n+1} s.t. $stp(a_{n+1}b_{n+1}a_n/\emptyset)=stp(aba_0/\emptyset)$ and $a_{n+1}b_{n+1}\perp a_0\ldots a_{n-1}b_1\ldots b_{n-1}\,|\,a_n$.
In particular, a_n weakly isolates $a_{n+1}b_{n+1}$. By transitivity of the notion of weak isolation and induction,
i) for each n, a_0 weakly isolates a_nb_n.
By a similar induction:
ii) for each n, $b_1,\ldots,b_n,a_n$ is an independent sequence.
($b_1\ldots b_n\perp a_{n+1}b_{n+1}\,|\,a_n$; by induction, $b_1\ldots b_n\perp a_n$ so by transitivity of forking $b_1\ldots b_n\perp a_{n+1}b_{n+1}\,|\,\emptyset$. Since $a_{n+1}\perp b_{n+1}$, $b_1,\ldots,b_{n+1},a_{n+1}$ is independent.)
By (i), a_0 weakly isolates b_n for each n. Since $tp(b_n/\emptyset)=tp(b/\emptyset)$ is not isolated, $a_0\not\perp b_n$ for each n. But by (ii), the sequence $b_1,b_2,\ldots$ is independent. This contradicts (FW).
The propery (P) clearly implies that if there is any non-isolated type p over $\emptyset$, then there are infinitely many non-isomorphic models, and in fact $Pr(p^{(n)})\approx Pr(p^{(m)}) \Rightarrow n=m$.

Observation 2 Let T be finitely controlled and $\aleph_0$-categorical. Then T is ω-stable.
Proof Suppose not. Then one can easily find a type p such that p has no Morley rank, but every forking extension of p does. p must be regular: every forking extension of p has Morley rank, and p is orthogonal to every such type: if $a\models p\,|\,B$, $a\not\perp c\,|\,B$ and $tp(c/B)$ has Morley rank, then by the minimality of p $tp(a/Bc)$ also has Morley rank, so by the additivity of

Morley rank for $\aleph_0$-categorical theories $p=tp(a/B)$ has Morley rank, contradiction. So p is regular.

By hypothesis, p is non-orthogonal to some finite set B. By the proof of Shelah's theorem on semi-regular types, there exists an element c such that $q=stp(c/B)$ is non-orthogonal to p, and satisfying the following: there exist $\hat{B}\supset B$, $c\models q|\hat{B}$, and a (finite) set I such that $c\in dcl(B\cup I)$, and for each $d\in I$, $stp(d/\hat{B})$ is a non-forking extension of a conjugate of p. Since every forking extension of p has Morley rank, it also has U-rank, so p itself has ordinal U-rank. Therefore $q|\hat{B}$ has U-rank, so $q|B$ has U-rank. But B is finite and T is $\aleph_0$-categorical, so q is isolated by some formula D. Now the relativized theory of our model inside D is superstable and $\aleph_0$-categorical, so by Lachlan's theorem is is ω-stable. In other words, q has Morley rank. This contradicts the observations made above that p is orthogonal to every type with Morley rank, and non-orthogonal to q

Lemma 3 Let T be a countable, finitely controlled stable theory. Assume that every sufficiently large model of T is $\aleph_0$-saturated. Then T is either $\aleph_0$-categorical, $\aleph_0$-stable or superstable, unidimensional.

Proof If T is ω-stable then it is easy to see that it must be ω-categorical. So assume it is not. Let **P** be the set of formulas with ordinal Morley rank. For every countable M, only countable many types over M are **P**-analyzable in finitely many steps; so there exists p such that p is not (finitely) **P**-analyzable. Recall that if $p\in S(X)$ is **P**-analyzable, then there exists a finite $X_0\subset X$ such that p is based on X_0, and $p|X_0$ is analyzable Also, if q is **P**-analyzable then so is every forking extension of q. Hence we may use Zorn's lemma to choose p minimal with respect to not being analyzable, i.e. so that every forking extension of p is $\mathbf{P}$-analyzable. In particular every forking extension of p has U-rank, hence so does p. As in the previous proposition, p is orthogonal to every $\mathbf{P}$-analyzable type. Since T is finitely controlled, p is non-orthogonal to some finite set A. By 2.1.6, there exists a p-internal strong type q based on A; in particular q has U-rank. Being non-orthogonal to p, q cannot be $\mathbf{P}$-analyzable. In particular, q does not have Morley rank. Let r be an extension of q of least possible U-rank with respect to not having Morley rank. Note that r is automatically finitely based. We may assume r is stationary. Since every forking extension of r has Morley rank, so would r if it were isolated; so it is not. Let s be a stationary extension of r over a finite set B of least possible U-rank with respect to not being isolated. So s is not isolated, but every forking extension of s is. Moreover the same is true of $s|C$ for any finite set C. Hence for any finite $C \supset B$, $s|C$ has a unique extension to the prime model Pr(C). It follows that every type over every finite set is non-orthogonal to s: Let $t \in S(C)$, C finite, $C \supset B$. Let M be an I-isolated model over $Pr(C) \cup I$, where I is a long Morley sequence in $t|M$. Then M is $\aleph_0$-saturated, so $s|C$ is realized in M. Any realization of $s|C$ must realize $s|Pr(C)$. Since every element of M forks with I over Pr(C), $s \not\perp t$. Since s is regular, all finitely based types are non-orthogonal. By the proof of Theorem 3.4.1, T is superstable. But then all types are finitely based, so T is unidimensional.

Corollary 4 (Strong Kueker Conjecture for finitely controlled theories.) Let T be a countable, finitely controlled stable theory. Then the following are equivalent:

(i) Every sufficiently large model of T is $\aleph_0$-saturated

(ii) T has a unique countable model with an infinite set of indiscernibles.

Proof If T has a model of cardinality $\beth_{\omega_1}$ omitting a type over a finite set, then an Ehrenfeucht-Mostowski model can be found omitting the same type, so there exists a countable model containing an infinite set of indiscernibles other that the saturated model. Thus (ii) always implies (i). Conversely, suppose (i) & $\neg$(ii). Let M be a countable model, $I \subset M$ an infinite indiscernible set, and suppose M is not the saturated model. By the lemma, T is superstable. Let p be a type based on a finite subset of M such that p is not realized in M, and choose p of least possible U-rank. Let q be the average of I. By superstable reflection, $p \perp^{a} q^{(\omega)}$. Thus just as in the lemma, p is not realized in any l-isolated model over $M \cup I$ where I is a Morley sequence of realizations of q. This contradicts (i).

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