Lecture #1

- complex numbers were introduced to give solutions to quadratic equations.
- For example,

$$x^2 + 1 = 0$$

has no solution in the real numbers, $\mathbb{R}$.

- to resolve this, we introduce i, imaginary unit, and define $i^2 = -1$.
- thus, i is a solution to $x^2+1=0$.
- the field of complex numbers $\mathbb{C} = \mathbb{R}[i]$ is the set of numbers of the form $\{a+bi : a,b\in\mathbb{R}\}$.
- Note that $\mathbb{R} \subseteq \mathbb{C}$.
- Fact (Fundamental Theorem of Algebra)
  Let $x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 = 0$ be a polynomial with $a_i \in \mathbb{C}$. Then it has n solutions in $\mathbb{C}$.

We usually denote complex numbers by z or w.

- Given $z = a + bi$,

$$\mathrm{Re}(z) = a \quad \text{is the "real part"}$$

$$\mathrm{Im}(z) = b \quad \text{is the "imaginary part".}$$

Ex: $\mathrm{Re}(2-4i) = 2$

$\mathrm{Im}(2-4i) = -4$.

So $z = \mathrm{Re}(z) + \mathrm{Im}(z)i$ for all z.

Fact: Given two complex numbers z and w, we have $z = w$ if and only if $\begin{cases} \mathrm{Re}(z) = \mathrm{Re}(w) \\ \mathrm{Im}(z) = \mathrm{Im}(w). \end{cases}$

i.e. $a+bi = c+di$ iff $a = c$ and $b = d$.

Arithmetic of Complex numbers.

Let $z = a+bi$, $w = c+di$. Then

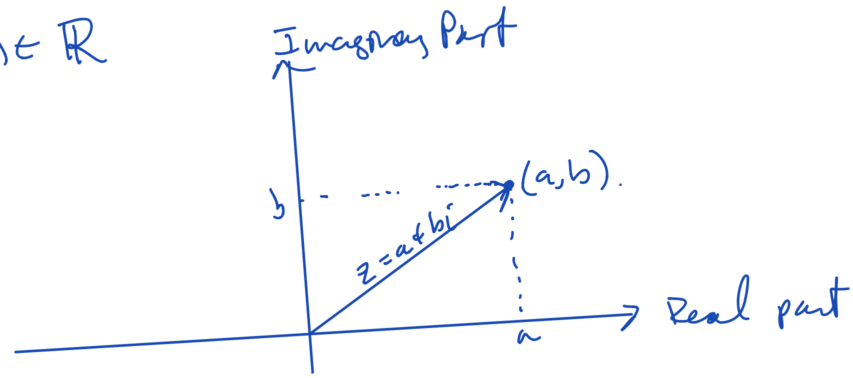
$$\begin{aligned} z + w &= (a+bi) + (c+di) \\ &= (a+b) + (b+d)i \end{aligned}$$

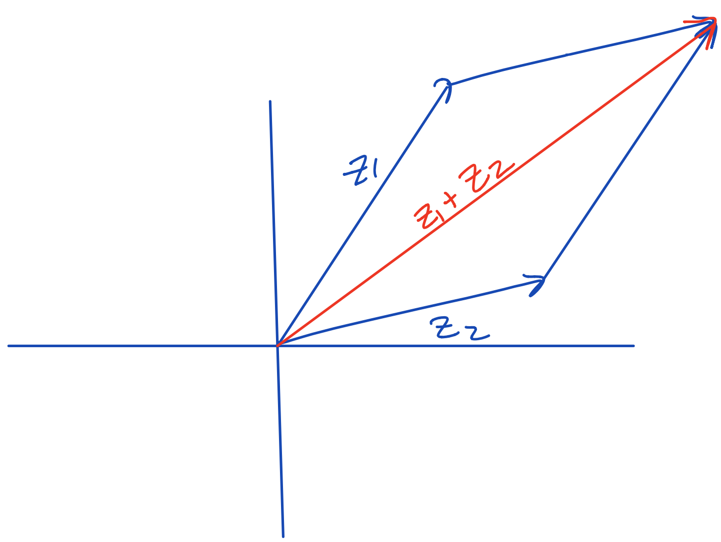
$$zw = (a+bi)(c+di)$$
$$= ac + adi + bci + bdi^2 \quad \text{(foil!)}$$
$$= ac + (ad+bc)i + bd(-1)$$
$$= (ac-bd) + (ad+bc)i$$

Special case: if $c \in \mathbb{R}$, then $cz = c(a+bi) = ac + bci$.

$$z - w = z + (-1)w = (a-c) + (b-d)i$$

To each complex number $a+bi$ we can associate a real vector $(a,b) \in \mathbb{R}$

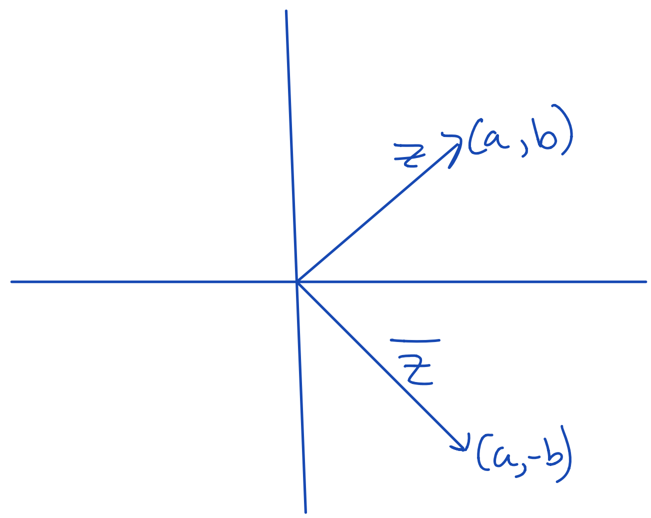




Complex conjugation

Given a complex number $z = a+bi$, the complex conjugate is defined to be

$$\bar{z} = a - bi$$



Note that $z = \bar{z}$ iff $z \in \mathbb{R}$ i.e. z is real.

- The modulus / absolute value of a complex number z is defined to be

$$\sqrt{z\bar{z}} = \left((a+bi)(a-bi)\right)^{1/2}$$
$$= \left(a^2 - b^2 i^2\right)^{1/2} = \left(a^2+b^2\right)^{1/2}.$$
$$= \sqrt{a^2+b^2}$$

- The modulus of z is the length of the vector (a,b)!

- If $z=a$ is a real number, then $|z|=|a|$.

Inverses: $\mathbb{C}$ is a field. That means we can add, multiply, and divide.

- For $z \in \mathbb{C}$, we define $z^{-1} = \left(\frac{1}{z}\right)$ to be the unique complex number such that $z\left(\frac{1}{z}\right)=1$.

- If $z=a+bi$ and $\frac{1}{z} = c+di$, we have

$$1=(a+bi)(c+di)=(ac-bd)+(ad+bc)i=1+0i$$

This gives two equations in two variables (c and d)

$$\begin{aligned} ac-bd &= 1 \\ ad+bc &= 0 \end{aligned}$$

which has a unique solution (exercise!)

$$c = \frac{a}{|z|^2} \qquad d = \frac{-b}{|z|^2}$$

which gives
$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}.$$

Therefore, given two complex numbers z, w,
$$\frac{z}{w} = z\left(\frac{1}{w}\right) = \frac{z\overline{w}}{|w|^2}$$

Theorem: For any complex numbers z, z_1, z_2,

1) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
2) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
3) $\overline{\left(1/z\right)} = \frac{1}{\overline{z}}$
4) $\overline{\overline{z}} = z$.
5) $|\overline{z}| = |z|$
6) $|z_1 z_2| = |z_1||z_2|$
7) $|z_1 + z_2| \le |z_1| + |z_2|$.