Lecture #10

Recall from last time:

A real (complex) inner prod. space is a real (resp. complex) vector space V together with a function $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$ (resp. $\mathbb{C}$) such that

1) $\langle \vec{v}, \vec{u} \rangle = \langle \vec{u}, \vec{v} \rangle$ (resp. $\langle \vec{v}, \vec{u} \rangle = \overline{\langle \vec{u}, \vec{v} \rangle}$
2) $\langle \vec{u}+\vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$
3) $\langle k\vec{v}, \vec{w} \rangle = k \langle \vec{v}, \vec{w} \rangle$.
4) $\langle \vec{v}, \vec{v} \rangle \geq 0$, $\langle \vec{v}, \vec{v} \rangle = 0$ iff $\vec{v} = \vec{0}$.

Last time: we saw that for $(V, \langle \cdot , \cdot \rangle)$ an inner product space, then for any $\vec{u}, \vec{v} \in V$, we have

$$|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \|\vec{v}\|$$

the Cauchy-Schwarz Inequality.

Theorem: Let $\vec{u}, \vec{v}$ be vectors in an inner product space. Then

1) $\|\vec{u}\| \geq 0$.
2) $\|\vec{u}\| = 0$ iff $\vec{u} = \vec{0}$.
3) $\|k\vec{u}\| = |k| \|\vec{u}\|$
4) (Triangle Inequality) $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$.

1) and 2) follow immediately from the axioms.

For 3)

$$\|k\vec{u}\| = \sqrt{\langle k\vec{u}, k\vec{u}\rangle} = \sqrt{k\langle \vec{u}, k\vec{u}\rangle} = \sqrt{k\,\overline{\langle k\vec{u}, \vec{u}\rangle}} = \sqrt{k\bar{k}\langle \vec{u}, \vec{u}\rangle}$$

$$= |k|\,\|\vec{u}\|.$$

4) $\|\vec{u}+\vec{v}\|^2 = \langle \vec{u}+\vec{v}, \vec{u}+\vec{v}\rangle$ (complex proof. Real case in text).

$$= \langle \vec{u}, \vec{u}+\vec{v}\rangle + \langle \vec{v}, \vec{u}+\vec{v}\rangle.$$

$$= \overline{\langle \vec{u}+\vec{v}, \vec{u}\rangle} + \overline{\langle \vec{u}+\vec{v}, \vec{v}\rangle}$$

$$= \overline{\langle \vec{u}, \vec{u}\rangle} + \overline{\langle \vec{v}, \vec{u}\rangle} + \overline{\langle \vec{u}, \vec{v}\rangle} + \overline{\langle \vec{v}, \vec{v}\rangle}.$$

$$= \langle \vec{u}, \vec{u}\rangle + \langle \vec{u}, \vec{v}\rangle + \overline{\langle \vec{u}, \vec{v}\rangle} + \langle \vec{v}, \vec{v}\rangle.$$

$$\leq \langle \vec{u}, \vec{u}\rangle + |\langle \vec{u}, \vec{v}\rangle| + |\langle \vec{u}, \vec{v}\rangle| + \langle \vec{v}, \vec{v}\rangle$$

$$= \langle \vec{u}, \vec{u}\rangle + 2|\langle \vec{u}, \vec{v}\rangle| + \langle \vec{v}, \vec{v}\rangle.$$

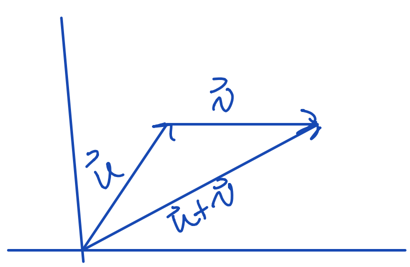
$$\leq \langle \vec{u}, \vec{u}\rangle + 2\|\vec{u}\|\|\vec{v}\| + \langle \vec{v}, \vec{v}\rangle \text{ by C.S.}$$

$$= \|u\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2$$

$$= (\|\vec{u}\| + \|\vec{v}\|)^2$$

Hence $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ $\square$

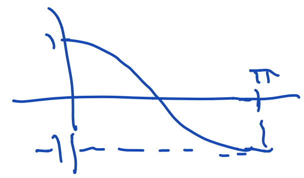
Picture



- From the CS inequality, for all $\vec{u}, \vec{v} \in V$,

$$-1 \leq \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|} \leq 1$$

- Now, $\cos(\theta)$ is injective on $[0, \pi]$



and so for any $\vec{u}, \vec{v} \in V$, there is a unique angle θ such that

$$\cos(\theta) = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$

- Therefore, given an inner product space $(V, \langle \cdot, \cdot \rangle)$, and vectors $\vec{u}, \vec{v} \in V$, we may define the angle between $\vec{u}$ and $\vec{v}$ as

$$\theta = \cos^{-1}\left(\frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|} \right).$$

Definition: Given an inner product space $(V, \langle \cdot, \cdot \rangle)$, and $\vec{u}, \vec{v} \in V$, we say that $\vec{u}$ and $\vec{v}$ are orthogonal iff $\langle \vec{u}, \vec{v} \rangle = 0$.

Note: that orthogonality depends on the inner product.

For example: take $\vec{u}=(1,1)$, $\vec{v}=(1,-1) \in \mathbb{R}^2$.

Then with respect to the dot product

$$\vec{u}\cdot\vec{v} = (1)(1)+(1)(-1) = 1-1 = 0,$$

so they are orthogonal. <u>but</u> if

$\langle \vec{u},\vec{v}\rangle = u_1v_1 + 2u_2v_2$, then

$$\langle \vec{u},\vec{v}\rangle = (1)(1) + 2(1)(-1) = 1-2 = -1$$

so not orthogonal.

<u>Ex:</u> Let $V = C([0,\pi])$ the real vector space of continuous functions on $[0,\pi]$.

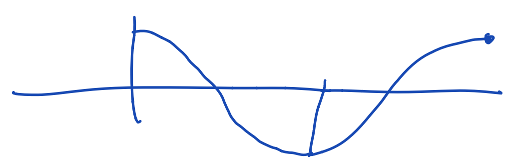
Let $\langle f,g\rangle = \int_0^{\pi} f(x)g(x)\,dx$.

Let $f(x) = \sin(x)$ $g(x) = \cos(x)$.

Then $\langle f,g\rangle = \int_0^{\pi} \sin(x)\cos(x)\,dx$.

Recall from a double angle identity, $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$.

So $\langle f,g\rangle = \int_0^{\pi} \frac{\sin(2x)}{2}\,dx$



$= \left.\frac{-\cos(2x)}{4}\right|_0^{\pi}$

$\frac{d}{dx} -\cos(2x) = \sin(2x)2.$

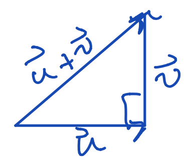
$$= \frac{-\cos(0)}{4} - \left(\frac{-\cos(2\pi)}{4}\right)$$

$$= -\frac{1}{4} + \frac{1}{4} = 0.$$

So $\sin(x)$ and $\cos(x)$ are orthogonal.

<u>Theorem</u> (Generalized Theorem of Pythagoras).
Let $\vec{u}, \vec{v} \in V$ be orthogonal. Then

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|v\|^2$$



<u>Proof:</u>

$$\|\vec{u}+\vec{v}\|^2 = \langle \vec{u}+\vec{v}, \vec{u}+\vec{v} \rangle = \langle \vec{u}, \vec{u} \rangle + \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{u} \rangle + \langle \vec{v}, \vec{v} \rangle$$

Since $\vec{u}, \vec{v}$ are orthogonal, $\langle \vec{u}, \vec{v} \rangle = 0 = \langle \vec{v}, \vec{u} \rangle$.

So

$$\|\vec{u}+\vec{v}\|^2 = \langle \vec{u}, \vec{u} \rangle + \langle \vec{v}, \vec{v} \rangle = \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

$\square$