Lecture # 11

Definition: Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Let $W \subseteq V$ be a subspace of V. The orthogonal complement of W, is the set of vectors orthogonal to every vector in W:

$$W^{\perp} = \{\vec{v} \in V : \langle \vec{v}, \vec{w} \rangle = 0 \text{ for all } \vec{w} \in W\}.$$

Theorem: Let $W \subseteq V$. Then:

1) $W^{\perp}$ is a subspace of V.
2) $W \cap W^{\perp} = \{\vec{0}\}$.

Pf: 1) $W^{\perp} \subseteq V$, so we only need to check that $W^{\perp}$ is closed under addition and scalar multiplication.

i) Addition. Let $\vec{v}_1, \vec{v}_2 \in W^{\perp}$. Let $\vec{w} \in W$ be arbitrary. Then

$$\langle \vec{v}_1 + \vec{v}_2, \vec{w} \rangle = \langle \vec{v}_1, \vec{w} \rangle + \langle \vec{v}_2, \vec{w} \rangle = 0 + 0 = 0.$$

ii) Scalar multiplication: Let k be a scalar, $\vec{v} \in W^{\perp}$. Then for $\vec{w} \in W$ arbitrary,

$$\langle k\vec{v}, \vec{w} \rangle = k\langle \vec{v}, \vec{w} \rangle = k \cdot 0 = 0.$$

2). Let $\vec{v} \in W \cap W^{\perp}$. Then $\langle \vec{v}, \vec{v} \rangle = 0$. By positivity, $\vec{v} = \vec{0}$.

Hence $W \cap W^{\perp} = \{\vec{0}\}$. $\square$

Fact (without Proof) Let $W \subseteq V$ be such that $\dim(W) < \infty$. Then $(W^{\perp})^{\perp} = W$

Remark In general this is not true. If $\dim(W)=\infty$, then we only have that $W \subseteq (W^\perp)^\perp$.

Connection to matrices

Theorem: Let A be an $m\times n$ matrix (real or complex).

1. Then $\ker(T_A) = \text{null}(A)$ is the orthogonal complement of the row space.
2. $\text{null}(A^T)$ is the orthogonal complement of the column space of A (not A^T!).

Gram-Schmidt and QR decomposition (6.3).

Defn: Let S be a set of vectors in an inner product space. We say that

i) S is <u>orthogonal</u> if $\forall \vec{v},\vec{w}\in S$, $\langle \vec{v},\vec{w}\rangle = 0$.

ii) S is <u>orthonormal</u> if S is orthogonal <u>and</u> $\|\vec{v}\| = 1$ for all $\vec{v}\in S$.

Ex: Let $\vec{v}_1 = (1,0,1)$, $\vec{v}_2 = (0,1,0)$, $v_3 = (1,0,-1)$.

Then $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is orthogonal wrt dot product, since

$\langle \vec{v}_1,\vec{v}_2\rangle = (1)(0)+(0)(1)+(1)(0) = 0$ $\quad$ $\langle \vec{v}_1,\vec{v}_3\rangle = (1)(1)+(0)(0)+(1)(-1)$
$= 1-1 = 0.$

$\langle \vec{v}_2, \vec{v}_3 \rangle = (0)(1) + (1)(0) + (0)(-1) = 0.$

- Often, especially for computational reasons, it is desireable to have an <u>orthonormal</u> set.
- Given an orthogonal set, we can get an orthonormal set by <u>normalizing</u> each vector.

<u>Fact:</u> Let $\vec{v}$ be any vector in an inner product space. Then $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$ is a unit vector.

<u>Pf:</u> $\|\vec{u}\| = \left\| \frac{\vec{v}}{\|\vec{v}\|} \right\| = \frac{1}{\|\vec{v}\|} \cdot \|\vec{v}\| = 1.$

<u>Ex:</u> Let $\vec{v}_1 = (1,0,1)$, $\vec{v}_2 = (0,1,0)$, $\vec{v}_3 = (1,0,-1)$.

Then an orthonormal set is

$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $\vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = (0,1,0)$

$\vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right).$

Note that rescaling does not change orthogonality. If $\langle \vec{u}, \vec{v} \rangle = 0$, then $\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle = 0$ for any scalar k.

Theorem: Let $\{\vec{v}_1, v_2, \dots \vec{v}_n\}$ be an orthogonal set of non-zero vectors. Then $\{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly independent.

Pf: Suppose

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots c_n\vec{v}_n = \vec{0}.$$

Let $1 \le i \le n$ be arbitrary. Then

$$0 = \langle \vec{0}, \vec{v}_i \rangle = \langle c_1\vec{v}_1 + \dots + c_n\vec{v}_n, \vec{v}_i \rangle$$

by linearity in first coord.

$$= \langle c_1\vec{v}_1, \vec{v}_i \rangle + \langle c_2\vec{v}_2, \vec{v}_i \rangle + \dots + \langle c_n\vec{v}_n, \vec{v}_i \rangle$$

$$= c_1\cancelto{0}{\langle \vec{v}_1, \vec{v}_i \rangle} + \dots + c_i\langle \vec{v}_i, \vec{v}_i \rangle + \dots + c_n \cancelto{0}{\langle \vec{v}_n, \vec{v}_i \rangle}$$

$$= c_i\langle v_i, v_i \rangle \quad \text{by orthogonality.}$$

Now, since $\vec{v}_i \neq \vec{0}$, $\langle \vec{v}_i, \vec{v}_i \rangle \neq 0$ by the positivity axiom. Hence $c_i = 0$. Since i was arbitrary,

$$c_1 = c_2 = \dots = c_n = 0. \quad \square.$$

Defn:
- We say that a basis is an orthonormal basis if the set of basis vectors is orthonormal.
- We say a basis is an orthogonal basis if all the vectors are orthogonal.

Ex: The standard basis
$\hat{e}_1 = (1, 0 \dots, 0)$, ..., $\hat{e}_n = (0, 0, \dots, 0, 1)$

is an orthonormal basis for K^n ($K = \mathbb{C}$ or $\mathbb{R}$).

- Recall that if $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for V, then any vector $\vec{u} \in V$ can be expressed uniquely as

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n.$$

- The typical way to find the coefficients is to solve the corresponding linear system

$$\vec{u} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

- However is S is orthogonal or orthonormal, its even easier: