Lecture #12

Recall: A basis B for an inner prod. space V is called:

i) orthogonal iff forall $\vec{v}, \vec{w} \in B$ if $\vec{v} \neq \vec{w}$ then $\langle \vec{v}, \vec{w} \rangle = 0$.

ii) orthonormal iff B is orthogonal and, forall $\vec{v} \in B$, $\|\vec{v}\| = 1$.

Theorem: Let $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for V and $\vec{u} \in V$.

i) If S is orthogonal then

$$\vec{u} = \frac{\langle \vec{u}, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 + \frac{\langle \vec{u}, \vec{v}_2 \rangle}{\|v_2\|^2} v_2 + \cdots + \frac{\langle \vec{u}, \vec{v}_n \rangle}{\|v_n\|^2} v_n$$

ii) If S is orthonormal, then

$$\vec{u} = \langle \vec{u}, \vec{v}_1 \rangle \vec{v}_1 + \cdots + \langle \vec{u}, \vec{v}_n \rangle \vec{v}_n.$$

Proof: i) Since $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis, we may uniquely write $\vec{u} = c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n$ for some scalars c_i. Thus, it suffices to show that $c_i = \frac{\langle \vec{u}, v_i \rangle}{\|v_i\|^2}$ for all $1 \leq i \leq n$. To see this, let i be arbitrary. Then

$$\langle u, v_i\rangle = \langle c_1\vec{v}_1 + \cdots + c_n\vec{v}_n, v_i\rangle$$
$$= c_1\langle \cancel{\vec{v}_1, \vec{v}_i}\rangle + \cdots + c_i\langle \vec{v}_i, \vec{v}_i\rangle + \cdots + c_n\langle \cancel{\vec{v}_n, \vec{v}_i}\rangle$$
$$= c_i\langle v_i, v_i\rangle = c_i\,\|\vec{v}_i\|^2.$$

Solving for c_i, $\quad c_i = \dfrac{\langle u, v_i\rangle}{\|v_i\|^2}.$

For ii) we use the same proof, but by normality $\|v_i\|^2 = 1$ and so $c_i = \langle u, v_i\rangle$.

$\square$

As a consequence of the last theorem, if $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is an orthogonal basis for V, and $\vec{u} \in V$, then the vector of coordinates of u is
$$(\vec{u})_S = \left(\frac{\langle \vec{u}, \vec{v}_1\rangle}{\|v_1\|^2}, \ldots, \frac{\langle \vec{u}, \vec{v}_n\rangle}{\|v_n\|^2}\right).$$

And if S is orthonormal, then
$$(\vec{u})_S = (\langle \vec{u}, \vec{v}_1\rangle, \ldots, \langle \vec{u}, \vec{v}_n\rangle).$$

Ex: Recall from last time that
$\vec{v}_1 = \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\rangle$, $\vec{v}_2 = \langle 0, 1, 0\rangle$, $\vec{v}_3 = \langle \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\rangle$.
is an orthonormal basis for $\mathbb{R}^3$. Let $u = (1,1,1)$ be a vector. Then we can write
$$\vec{u} = \left(\tfrac{1}{\sqrt{2}} + 0 + \tfrac{1}{\sqrt{2}}\right)\vec{v}_1 + (0 + 1 + 0)\,\vec{v}_2 + \left(\tfrac{1}{\sqrt{2}} + 0 - \tfrac{1}{\sqrt{2}}\right)\vec{v}_3$$

$$= \frac{2}{\sqrt{2}}\vec{v}_1 + \vec{v}_2$$

So $(\vec{u})_S = \left(\frac{2}{\sqrt{2}}, 1, 0\right)$.

Producing orthogonal Bases

- For many applications we want an orthogonal or orthonormal basis.
- We want to give a method for producing an orthogonal basis. To do this we need to introduce

Theorem (Projection Theorem). If W is a finite dimensional subspace of an I.P.S. V, then every $\vec{v} \in V$ can be expressed uniquely as

$$\vec{v} = \vec{w}_1 + \vec{w}_2$$

where $\vec{w}_1 \in W$ and $\vec{w}_2 \in W^\perp$.

- These vectors are usually denoted as

$\vec{w}_1 = \text{proj}_W \vec{v}$ the orthogonal projection of $\vec{v}$ onto W.

and $\vec{w}_2 = \text{proj}_{W^\perp} \vec{v}$ the orthogonal projection of $\vec{v}$ onto $W^\perp$.

- These projections can be computed via the following theorem:

Theorem: Let W be a finite dimensional subspace of an I.P.S. V.

a) If $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthogonal basis for W and $\vec{u}$ is any vector in V, then

$$\text{proj}_W \vec{u} = \frac{\langle \vec{u}, \vec{v}_1 \rangle}{\|v_1\|^2}\vec{v}_1 + \dots + \frac{\langle \vec{u}, \vec{v}_n \rangle}{\|\vec{v}_n\|^2}\vec{v}_n$$

b) If $\{\vec{v}_1, \dots, \vec{v}_n\}$ is orthonormal basis for W, the

$$\text{proj}_W \vec{u} = \langle \vec{u}, \vec{v}_1 \rangle \vec{v}_1 + \dots + \langle \vec{u}, \vec{v}_n \rangle \vec{v}_n.$$

Pf: Let $\vec{u} \in V$. By the projection theorem, we can write $\vec{u} = \vec{w}_1 + \vec{w}_2$ where

$$\vec{w}_1 = \text{proj}_W \vec{u}$$
$$\vec{w}_2 = \text{proj}_{W^\perp} \vec{u}.$$

By a previous result, we have

$$\text{proj}_W \vec{u} = \vec{w}_1 = \frac{\langle \vec{w}_1, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 + \dots + \frac{\langle \vec{w}_1, \vec{v}_n \rangle}{\|\vec{v}_n\|^2}\vec{v}_n.$$

Now, since $\vec{w}_2 \in W^\perp$, we have

$$\langle \vec{w}_2, \vec{v}_1 \rangle = \dots = \langle \vec{w}_2, \vec{v}_n \rangle = 0,$$

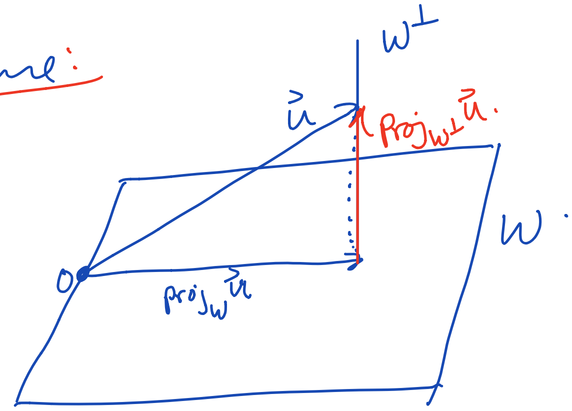
and so, by linearity in the first variable,

$$\text{proj}_W \vec{u} = \frac{\langle \vec{w}_1 + \vec{w}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 + \dots + \frac{\langle \vec{w}_1 + \vec{w}_2, \vec{v}_n \rangle}{\|\vec{v}_n\|^2}\vec{v}_n$$

$$= \frac{\langle \vec{u}, \vec{v}_1 \rangle}{\|\vec{v}_1\|}\vec{v}_1 + \dots + \frac{\langle \vec{u}, \vec{v}_n \rangle}{\|v_n\|^2}\vec{v}_n$$

as required.

Picture:



Ex: Consider $\mathbb{R}^3$ with the dot product.
Let $\underbrace{\vec{v}_1 = (0,1,0) \quad \vec{v}_2 = \left(\frac{-4}{5}, 0, \frac{3}{5}\right)}_{\text{orthonormal.}}$. Let $W = \text{Span}\{\vec{v}_1, \vec{v}_2\}$.

Let $\vec{u} = (1,1,1)$. Then

$$\text{proj}_W \vec{u} = (\vec{u} \cdot \vec{v}_1)\vec{v}_1 + (\vec{u} \cdot \vec{v}_2)\vec{v}_2$$
$$= (0+1+0)\vec{v}_1 + \left(\frac{-4}{5} + 0 + \frac{3}{5}\right)\vec{v}_2 = \vec{v}_1 + \frac{-1}{5}\vec{v}_2$$
$$= (0,1,0) + \left(\frac{4}{25}, 0, \frac{-3}{25}\right) = \left(\frac{4}{25}, 1, \frac{-3}{25}\right)$$

So, how to find $\text{proj}_{W^{\perp}} \vec{u}$? Since we know that $\vec{u} = \text{proj}_W \vec{u} + \text{proj}_{W^{\perp}} \vec{u}$, hence

$$\text{proj}_{W^{\perp}} u = \vec{u} - \text{proj}_W \vec{u}$$
$$= (1,1,1) - \left(\frac{4}{25}, 1, \frac{-3}{25}\right)$$
$$= \left(\frac{21}{25}, 0, \frac{28}{25}\right).$$

Theorem: Let V be a finite dimensional inner product space. Then V has an orthogonal basis (and hence an orthonormal basis, by normalizing).

Proof: Next time!