Lecture 13

Recall from last time:

Projection theorem: Let $W \subseteq V$ be a finite dimensional subspace of an inner product space V. Then for all $\vec{v} \in V$, we can write

$$\vec{v} = \vec{w}_1 + \vec{w}_2$$

where $\vec{w}_1 = \text{proj}_W \vec{v} \in W$
and $\vec{w}_2 = \text{proj}_{W^\perp} \vec{v} = v - \text{proj}$

and

Theorem: Let W be a finite dim subspace of an inner product space V. If $\{\vec{u}_1, \dots, \vec{u}_n\}$ is an orthogonal basis for W, then for any $\vec{v} \in V$,

$$\text{proj}_W \vec{v} = \frac{\langle \vec{v}, \vec{w}_1 \rangle}{\|w_1\|^2} \vec{w}_1 + \dots + \frac{\langle \vec{v}, \vec{w}_n \rangle}{\|w_n\|^2} \vec{w}_n$$

(and so $\text{proj}_{W^\perp} \vec{v} = \vec{v} - \left(\frac{\langle \vec{v}, \vec{w}_1 \rangle}{\|\vec{w}_1\|^2} \vec{w}_1 + \dots \frac{\langle \vec{v}, \vec{w}_n \rangle}{\|\vec{w}_n\|^2} \vec{w}_n \right)$)

- we use these theorems recursively to prove the following:

Theorem: Let V be a finite dimensional inner product space. Has an orthogonal basis.

Proof: • Let $\{\vec{u}_1, \dots, \vec{u}_n\}$ be a (any) basis for V (we have a finite basis since V is finite dimensional).

• From $\{\vec{u}_1, \dots \vec{u}_n\}$ we want to produce an orthogonal basis $\{\vec{v}_1, \dots, \vec{v}_n\}$.

• We construct this basis in n steps.

Step 1: Let $\vec{v}_1 := \vec{u}_1$.

Step 2: We want a vector $\vec{v}_2$ orthogonal to $\vec{v}_1$. Let $W_1 = \text{span}(\vec{v}_1)$. Define

$$\vec{v}_2 := \text{proj}_{W_1^\perp} \vec{u}_2 = \vec{u}_2 - \text{proj}_{W_1} \vec{u}_2$$
$$= \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|v_1\|^2} \vec{v}_1$$

* Note! $\vec{v}_2 \neq \vec{0}$, since otherwise $\vec{u}_1$ and $\vec{u}_2$ would be linearly dependent, which is impossible!

Step 3: We want a vector $\vec{v}_3$ which is orthogonal to $\vec{v}_1$ and $\vec{v}_2$. Let $W_2 = \text{span}(\vec{v}_1, v_2)$. Note that $\{\vec{v}_1, \vec{v}_2\}$ is an orthogonal basis of W_2. Define

$$\vec{v}_3 := \text{Proj}_{W_2^\perp} \vec{u}_3 = \vec{u}_3 - \text{Proj}_{W_2} u_3$$
$$= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|v_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|v_2\|^2} \vec{v}_2$$

As before $\vec{v}_3 \neq \vec{0}$, since $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are linearly indep.

<u>Step 4:</u> We want a vector $\vec{v}_4$ orthogonal to $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Let $W_3 = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$. Then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis for W_3. Define

$$\vec{v}_4 := \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3.$$

$\vdots$

<u>Step n:</u> Finally, let $W_{n-1} = \text{Span}(\vec{v}_1, \dots, \vec{v}_{n-1})$.

Take $\vec{v}_n = \vec{u}_n - \left(\sum_{i=1}^{n-1} \frac{\langle \vec{u}_n, v_i \rangle}{\|v_i\|^2} \right)$.

Then $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthogonal basis for V.

<u>Optional Step:</u> For $1 \leq i \leq n$, let

$$\vec{s}_i := \frac{\vec{v}_i}{\|\vec{v}_i\|}$$

then $\{\vec{s}_1, \vec{s}_2, \dots, \vec{s}_n\}$ is an orthonormal basis for V. $\square$.

- The proof is an algorithm called <u>Gram-Schmidt</u>. (G.S.).

<u>Ex:</u> Let $P_3(\mathbb{R})$ be the real inner product space of polynomials with real coefficients of degree at most 3 polynomials with real coefficients. We can equip $P_3(\mathbb{R})$ with the inner product

$$\langle p, q\rangle = \int_{-1}^{1} p(x)\,q(x)\,dx.$$

Observe that the standard basis $\{1, x, x^2, x^3\}$ is not orthogonal w.r.t. to this innerproduct

use G.S. to find an orthogonal basis:

<u>Step 1:</u> $\vec{v}_1 = 1.$

<u>Step 2:</u> $\vec{v}_2 = x - \dfrac{\langle x, 1\rangle}{\|1\|^2}\cdot 1$

$$= x - \frac{\int_{-1}^{1} x\,dx}{\int_{-1}^{1} 1^2\,dx}\,1 = x - \frac{\overset{0}{\cancel{\frac{x^2}{2}\Big|_{-1}^{1}}}}{2}$$

$$= x$$

<u>Step 3:</u> $\vec{v}_3 = x^2 - \dfrac{\langle x^2, 1\rangle}{\|1\|^2}\,1 - \dfrac{\langle x^2, x\rangle}{\|x\|^2}\,x$

$$= x^2 - \frac{\int_{-1}^{1} x^2\,dx}{\int_{-1}^{1} 1\,dx}\,1 - \frac{\overset{0.}{\cancel{\int_{-1}^{1} x^2\,x\,dx}}}{\int_{-1}^{1} x^2}\,x$$

$$= x^2 - \frac{\frac{x^3}{3}\Big|_{-1}^{1}}{2}$$

$$= x^2 - \frac{2/3}{2} = x^2 - \frac{1}{3}$$

<u>Step 4:</u>

$$\vec{v}_4 = x^3 - \frac{\overset{0}{\cancel{\int_{-1}^{1} x^3\,dx}}}{\int_{-1}^{1} 1\,dx} - \frac{\int_{-1}^{1} x^3\,x\,dx}{\int_{-1}^{1} x^2}\,x - \frac{\int_{-1}^{1} x^3(x^2 - 1/3)\,dx}{\int_{-1}^{1} (x^2 - 1/3)^2\,dx}\,(x^2 - 1/3)$$

$$= x^3 - \frac{(2/5)}{(3/2)}x - \frac{\cancel{\left.\left(\frac{x^6}{6} - \frac{x^4}{12}\right)\right|_{-1}^{1}}^{\,0}}{\left.\left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right)\right|_{-1}^{1}}$$

$$= x^3 - \frac{4}{15}x$$

<u>Exercise:</u> Normalize these polynomials.

<u>Note</u> that if you apply G.S. to an orthogonal set $\{\vec{v}_1, \dots, \vec{v}_n\}$ then it returns the same set.

<u>Note:</u> that we can combine G.S. with a previous result to prove:

<u>Theorem</u>: Let $W \subseteq V$ be a finite dimensional subspace of an inner product V, then every orthogonal set of nonzero vectors in W can be enlarged to an orthogonal basis for W.

<u>Pf:</u> Let $S = \{\vec{v}_1, \dots, v_n\} \subseteq W$ be an orthogonal set of nonzero vectors. Then S must be linearly independent. Therefore, we can expand S to a new set

$S' = \{\vec{v}_1, \ldots \vec{v}_n, \vec{w}_1, \ldots, \vec{w}_s\}$

which is linearly independent. (Plus/minus Theorem).

We may assume S' is a basis for W.

Applying G.S. to S' gives an orthogonal basis for W. $\square$