Lecture 16

Function Approximation: Fourier Series.

- Consider the following approximation problem:
  Given a function $f \in C[a,b]$, find the "best approximation to f using only functions from a specified subspace of $C[a,b]$.
- Recall that $C[a,b]$ is a real inner product space with an inner product defined by
  $$\langle f,g\rangle = \int_a^b f(x)g(x)\,dx.$$
- For two functions f and $g \in C[a,b]$, the <u>mean square error</u> is
  $$MSE = \langle f-g, f-g\rangle = \int_a^b [f(x)-g(x)]^2 dx$$
  $$= \|f-g\|^2.$$
- So our approximation can be restated in terms of a least squares approximation problem:

<u>Least Squares Approx:</u> Let $f \in C[a,b]$ and let $W \subseteq C[a,b]$ be a finite dimensional subspace. We want to find $g \in W$ that minimizes

$$\|f-g\|^2 = \int_a^b [f(x)-g(x)]^2 dx.$$

- It turns out that, even in this context, the best approximation theorem remains valid.

<u>Theorem:</u> Let $W \subseteq C[a,b]$ be a finite dimensional subspace, $f \in C[a,b]$. Then the function $g \in W$ that minimizes $\|f-g\|^2$ is $g = \text{proj}_W f$. (the least squares approx to f from W).

<u>Specific Example</u> Fourier Series.

- We consider $C[0,2\pi]$.
- we can easily compute projections when we have an orthogonal/orthonormal set.

<u>Fact from Calculus</u>: Let $f: \mathbb{R} \to \mathbb{C}$ be a complex valued function of a real variable. Write $f(x) = f_1(x) + i f_2(x)$. Then we define $\int_a^b f(x)dx$ as

$$\int_a^b f(x)dx = \int_a^b f_1(x)dx + i\int_a^b f_2(x)dx.$$

- Observe that, for $n \neq 0$,

$$\int_0^{2\pi} e^{inx} dx = \int_0^{2\pi} \cos(nx)dx + i\int_0^{2\pi} \sin(n\pi)dx = 0 + 0i = 0.$$

and $\int_0^{2\pi} e^{inx}dx = 2\pi$ if $n=0$.

- Suppose $m+n\neq 0$ $n,m\geq 0$, then

$$0 = \int_0^{2\pi} e^{inx} e^{imx} dx = \int_0^{2\pi} \cos(nx)\cos(mx) - \sin(nx)\sin(mx)dx$$

$$(*) \qquad + i\int_0^{2\pi} \cos(nx)\sin(mx) - \sin(nx)\cos(mx)dx.$$

- From the imaginary part of $(*)$

$$\int_0^{2\pi} \cos(nx)\sin(mx) - \sin(nx)\cos(mx)dx = 0$$

- substituting $-n$ for n, we get (for $m-n\neq 0$.).

$$\int_0^{2\pi} \cos(nx)\sin(mx) + \sin(nx)\cos(mx)dx = 0.$$

- Adding these together gives

$$\int_0^{2\pi} \cos(nx)\sin(mx)dx = 0 \quad \text{for } m\neq n.$$

Similarly considering the real part of $(*)$,

we get

$$\int_0^{2\pi} \cos(nx)\cos(mx)\,dx = 0 = \int_0^{2\pi} \sin(nx)\sin(mx) = 0.$$

For $m=-n$, we have

$$2\pi = \int_0^{2\pi} e^{nix} e^{-nix} dx = \int_0^{2\pi} (\cos(nx) + i\sin(nx))(\cos(nx) - i\sin(nx)) dx$$

$$= \int_0^{2\pi} \cos^2(nx) + \sin^2(nx) dx$$

- From calculus,

$$\int_0^{2\pi} \cos^2(nx) dx = \int_0^{2\pi} \sin^2(nx) dx$$

So $\displaystyle\int_0^{2\pi} \cos^2(nx) dx = \int_0^{2\pi} \sin^2(nx) dx = \pi$.

Conclusion:

$\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots, \cos(nx), \sin(nx), \ldots\}$

forms an orthogonal set in $C[0, 2\pi]$, and

$\|1\| = \sqrt{2\pi}$, $\|\cos(nx)\| = \|\sin(nx)\| = \sqrt{\pi}$.

Let $W_n \subseteq C[0, 2\pi]$ be the finite dimensional subspace spanned by

$\{1, \cos(x), \sin(x), \ldots, \cos(nx), \sin(nx)\}$.

- Since these functions are orthogonal it is easy to compute projections:

Let $f \in C[0, 2\pi]$. Then

$$\text{Proj}_{W_n} f = a_0 + (a_1 \sin(x) + \dots + a_n \sin(nx)) + (b_1 \cos(x) + \dots + b_n \cos(nx))$$

where
$$a_0 = \frac{\langle f, 1 \rangle}{\|1\|^2} = \frac{\langle f, 1 \rangle}{2\pi}$$

$$a_k = \frac{\langle f, \sin(kx) \rangle}{\|\sin(kx)\|^2} = \frac{\langle f, \sin(kx) \rangle}{\pi}.$$

$$b_k = \frac{\langle f, \cos(kx) \rangle}{\|\cos(kx)\|^2} = \frac{\langle f, \cos(kx) \rangle}{\pi}$$

<u>Ex:</u> Let $f(x) = x \notin W_n$.

Then $\text{proj}_{W_n}(x) = \pi - 2\left(\sin(x) + \frac{\sin(2x)}{2} + \dots + \frac{\sin(nx)}{n}\right).$

<u>Theorem:</u> Let $f \in C[0, 2\pi]$. Then

$$a_0 + a_1 \sin(x) + a_2 \sin(2x) + \dots + b_1 \cos(x) + b_2 \cos(2x) + \dots$$

converges to $f(x)$ with respect to $\|\cdot\|$

i.e. $\|f - \text{proj}_{W_n} f\| \to 0$ as $n \to \infty$.

Remark: Let $W \subseteq C[0,2\pi]$ be

$\operatorname{span}\{1, \cos(nx), \sin(nx) \mid n=1,2,\dots\}$.

By the previous theorem, $W^{\perp} = \{0\}$,

but $\{0\}^{\perp} = C[0,2\pi]$, so

$$W^{\perp\perp} = C[0,2\pi].$$

However $W \neq C[0,2\pi]$, since $x \notin W$.

Hence $W \subsetneq W^{\perp\perp}$.