Lecture 17.

We are moving on to chapter 8.

General linear Transformations.

- Recall that for an $m \times n$ matrix A, we have a map
$$T_A : \mathbb{R}^n \to \mathbb{R}^m$$
defined by $T_A(\vec{v}) = A\vec{v}$. which has the the following linearity properties:
$$T_A(\vec{u}+\vec{v}) = T_A(\vec{u}) + T_A(\vec{v}), \quad T_A(k\vec{v}) = kT(\vec{v})$$

- In fact, for any map $T : \mathbb{R}^n \to \mathbb{R}^m$ satisfying these linear properties there is an $m \times n$ matrix such that $T = T_A$. (How to do this?).

- From here we can define more general linear transformations.

<u>Definition</u>: Let V, W be V spaces. A map $T : V \to W$ is called a <u>linear map</u> iff it satisfies the following two properties for all $\vec{u}, \vec{v} \in V$ and all scalars k:

1) (Homogeneity) $T(k\vec{u}) = kT(\vec{u})$.

2) (Additivity) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$.

If $W = V$, then $T: V \to V$ is called a linear operator on V.

- Note that, combining these two properties

$$T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1T(\vec{v}_1) + \cdots + c_nT(\vec{v}_n).$$

Theorem: Let T be a linear transformation.

Then
1) $T(\vec{0}) = \vec{0}$.
2) $T(\vec{v}-\vec{u}) = T(\vec{v}) - T(\vec{u})$.

Pf: Easy. $\square$

Examples:

i) matrix transforms.

ii) The zero map: $T(\vec{v}) = \vec{0}$ $\forall \vec{v}$.

iii) The map $T: P_n \to P_{n+1}$ defined by

$$T(p(x)) = xp(x).$$

is linear.

iv) Let $\vec{v} \in \mathbb{R}^n$. Define a map $\mathbb{R}^n \to \mathbb{R}$ by

$$\langle \cdot, \vec{v} \rangle : \mathbb{R}^n \to \mathbb{R}$$
$$: \vec{u} \longmapsto \langle \vec{u}, \vec{v} \rangle.$$

(maps from $\mathbb{R}^n \to \mathbb{R}$ are called linear functionals).

Theorem: Let $T: V \to W$. Let $S = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis of V. Then for any $\vec{v} \in V$, we can express the image as

$$T(\vec{v}) = c_1 T(\vec{v}_1) + \dots + c_n T(\vec{v}_n)$$

for some constants $c_1 \dots c_n$.

In otherwords, T is determined by how it acts on a basis!

Proof: use linearity and properties of bases. $\square$

Ex: Let $V = C^1(-\infty, \infty)$ the space of continuously differentiable functions. Then

$$D: V \to C^0(-\infty, \infty)$$
$$: f(x) \mapsto \frac{d}{dx} f(x).$$

is a linear map.

Definition: Let $T: V \to W$ be a linear map.

The kernel of T is

$$\ker(T) = \{\vec{v} \in V : T(\vec{v}) = \vec{0}\}.$$

The range of T is the set

$$R(T) = \{\vec{w} \in W \mid (\exists \vec{v} \in V) T(\vec{v}) = \vec{w}\}.$$

Ex: What is $\ker(D)$, $D: C^1(-\infty,\infty) \to C(-\infty,\infty)$

($D(f)=0$, f is differentiable, so $f = c$ a constant).

Theorem Let $T: V \to W$ be a linear map.

1) $\ker(T)$ is a subspace of V.
2) $R(T)$ is a subspace of W.

Pf: 1). $\ker(T) \subseteq V$, so we need only to check that its closed under addition and scalar multiplication.

↳ Let $\vec{u}, \vec{v} \in \ker(T)$. Then

$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$$
$$= \vec{0} + \vec{0} = \vec{0}.$$

so $\vec{u}+\vec{v} \in \ker(T)$.

↳ Let $\vec{v} \in \ker(T)$, k a scalar. Then

$$T(k\vec{v}) = kT(\vec{v}) = k\vec{0} = \vec{0}.$$

so $k\vec{v} \in \ker(T)$.

Hence $\ker(T)$ is a subspace.

2) $R(T) \subseteq W$, so we only need to check that $R(T)$ is closed under addition and scalar multiplication.

↳ Let $\vec{w}_1, \vec{w}_2 \in R(T)$.

↳ By definition, there are $\vec{v}_1, \vec{v}_2 \in V$ such that $T(\vec{v}_1) = \vec{w}_1$ and $T(\vec{v}_2) = w_2$.

↳ then

$$\vec{w}_1 + \vec{w}_2 = T(\vec{v}_1) + T(\vec{v}_2) \quad \text{by linearity}$$
$$= T(\vec{v}_1 + \vec{v}_2) \in R(T).$$

↳ Let $\vec{w} \in R(T)$, k a scalar.

↳ since $\vec{w} \in R(T)$ there is $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$.

↳ Then

$$k\vec{w} = kT(\vec{v}) \quad \text{by linearity}$$
$$= T(k\vec{v}) \in R(T).$$

So $R(T) \subseteq W$ is a subspace.

$\square$.

<u>Definition</u>: Let $T: V \to W$. If $\ker(T)$ is finite dimensional, then the <u>nullity</u> of T is defined to be

$$\text{nullity}(T) = \dim(\ker(T)).$$

If $R(T)$ is finite dimensional, then the <u>rank</u> of T is defined to be

$$\text{rank}(T) = \dim(R(T)).$$

Important Theorem: (Rank-Nullity).

Let $T: V \to W$ be a linear map with $\dim(V) < \infty$. (W may have arbitrary dimension).

Then $$\dim(V) = \text{rank}(T) + \text{nullity}(T).$$

Pf: Next time!