Lecture 18.

Recap:

- Let $K = \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$.
- Let V, W be K-vectorspaces
- A map $T: V \to W$ is a <u>$(K$-$)$ linear map</u> if
  i) $\forall \vec{v}_1, \vec{v}_2 \in V$, $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$.
  ii) $\forall k \in K$, $T(k\vec{v}) = kT(\vec{v})$.
- The <u>kernel of T</u> is the set
  $$\ker(T) = \{\vec{v} \in V : T(\vec{v}) = \vec{0}\}.$$
  ↳ $\ker(T)$ is a subspace of V.
  ↳ The the nullity of T is
  $$\text{nullity}(T) = \dim(\ker(T)).$$
- The <u>range of T</u> is the set
  $$R(T) = \{\vec{w} \in W : \exists \vec{v} \in V (T(\vec{v}) = \vec{w}\}.$$
  ↳ $R(T)$ is a subspace of W.
  ↳ the rank of T is
  $$\text{rank}(T) = \dim(R(T)).$$

Theorem (Rank-Nullity Theorem).
Let V be finite dimensional, and let
$$T: V \to W$$
be a linear map. Then
$$\dim(V) = \text{rank}(T) + \text{nullity}(T).$$

Proof: Let $\dim(V) = n$.

If $\text{nullity}(T) = n$, then $\ker(T) = V$ and so $R(T) = \{\vec{0}\}$, so $\text{rank}(T) = 0$. Hence
$$n = \underbrace{\text{nullity}(T)}_{n} + \underbrace{\text{rank}(T)}_{=0}.$$

So suppose $0 \leq \text{nullity}(T) < n$. Let $i = \text{nullity}(T)$, and let S be a basis for $\ker(T)$, so $|S| = i$. (so $0 \leq i \lneq n$). $S = \{\vec{v}_1, \dots, \vec{v}_i\}$.

- By the Plus/Minus theorem, there are $n - i$ vectors $\vec{v}_{i+1}, \dots, \vec{v}_n \in V$ such that
$$S' = S \cup \{\vec{v}_{i+1}, \dots, \vec{v}_n\}$$
is a basis for V.
- It suffices to show that
$$\{T(\vec{v}_{i+1}), \dots, T(\vec{v}_n)\}$$
forms a basis for $R(T)$. Then

$$\dim(V) = n = i + (n-i)$$

- First, check that $\{T(\vec{v}_{i+1}), \dots, T(\vec{v}_n)\}$ spans $R(T)$.
  - ↳ suppose $\vec{w} \in R(T)$. Then there is $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$.
  - ↳ since S' is a basis for V, we can write

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_i\vec{v}_i + c_{i+1}\vec{v}_{i+1} + \cdots + c_n\vec{v}_n.$$

  - ↳ hence

$$\vec{w} = T(\vec{v}) = c_1 T(\vec{v}_1) + \cdots + c_i T(\vec{v}_i) + c_{i+1} T(\vec{v}_{i+1}) + \cdots + c_n T(\vec{v}_n).$$

  - ↳ since $\vec{v}_1, \dots, \vec{v}_i \in \ker(T)$

$$w = c_{i+1} T(v_{i+1}) + \cdots + c_n T(\vec{v}_n). \checkmark$$

- Next, check that $\{T(\vec{v}_{i+1}), \dots, T(\vec{v}_n)\}$ is linearly independent.
  - ↳ suppose

$$c_{i+1} T(\vec{v}_{i+1}) + \cdots + c_n T(\vec{v}_n) = \vec{0} \in W.$$

  - ↳ since T is linear

$$T(c_{i+1}\vec{v}_{i+1} + \cdots + c_n\vec{v}_n) = \vec{0}$$

    in other words,

$$c_{i+1}\vec{v}_{i+1} + \cdots + c_n\vec{v}_n \in \ker(T).$$

  - ↳ since S is a basis for $\ker(T)$, we can write

$$c_1\vec{v}_1+\cdots+c_i\vec{v}_i = c_{i+1}\vec{v}_{i+1}+\cdots+c_n\vec{v}_n$$

and so

$$c_1\vec{v}_1+\cdots+c_i\vec{v}_i - c_{i+1}\vec{v}_{i+1}-\cdots-c_n\vec{v}_n = \vec{0}.$$

↳ Since $\{\vec{v}_1,\ldots,\vec{v}_n\}$ is a basis for V, it is linearly independent, and hence

$$c_1 = c_2 = \cdots = c_n = 0.$$

↳ so $c_{i+1} = \cdots = c_n = 0$.

- Hence $\{T(v_{i+1}),\ldots,T(v_n)\}$ is a basis for $R(T)$ and so $\text{nullity}(T) = i$, $\text{rank}(T) = n-i$. $\square$.

8.2 Compositions and Inverses.

Let $T: V \to W$ be a linear transformation

<u>Defn</u>: T is one-to-one (injective) if it maps distinct vectors in V to distinct vectors in W, i.e. if $\vec{v}_1 \neq \vec{v}_2$ then $T(\vec{v}_1) \neq T(\vec{v}_2)$. Equivalently, T is injective iff $T(\vec{v}_1)=T(\vec{v}_2) \Rightarrow \vec{v}_1=\vec{v}_2$.

<u>Defn</u>: T is onto (aka surjective) if $R(T)=W$. i.e. everything "gets hit" by T.

<u>Theorem</u>: Let $T: V \to W$. Then T is injective/one-to-one. $\ker(T)=\vec{0}$.

Pf: Suppose T is injective. Let $\vec{v} \in \ker(T)$. Then $T(\vec{v}) = \vec{0}$.
Also $T(\vec{0}) = 0$. Since T is injective, $\vec{v} = \vec{0}$. ✓

On the other hand suppose that $\ker(T) = \{\vec{0}\}$. Let $\vec{v}_1, \vec{v}_2 \in V$
and suppose that $T(\vec{v}_1) = T(\vec{v}_2)$.

- Then $T(\vec{v}_1) - T(\vec{v}_2) = \vec{0}$.
- By linearity $T(\vec{v}_1 - \vec{v}_2) = \vec{0}$, so $\vec{v}_1 - \vec{v}_2 \in \ker(T)$.
- Since $\ker(T) = \{\vec{0}\}$, $\vec{v}_1 - \vec{v}_2 = \vec{0}$. Hence $\vec{v}_1 = \vec{v}_2$. So
  T is one-to-one. $\square$.

Theorem: Let $T: V \to W$ be linear. Suppose that V and W are finite dimensional with the same dimension.
Then TFAE:

1) T is 1-1/injective.
2) $\ker(T) = \{\vec{0}\}$
3) T is onto/surjective.

Pf: We already know that 1) and 2) are equivalent. So we just need to check that 2) and 3) are equivalent:

Let $\dim(V) = \dim(W) = n$.

↳ suppose $\ker(T) = \{\vec{0}\}$. Then $\text{nullity}(T) = 0$. By the rank-nullity theorem

$$n = \dim(V) = 0 + \text{rank}(T).$$

↳ so $\text{rank}(T) = n$, so $\dim(R(T)) = n$.

↳ since $R(T) \subseteq W$ and $\dim(W) = n = \dim(R(T))$, it follows that
$R(T) = W$. ✓

↳ On the otherhand, suppose T is onto, i.e. $R(T) = W$.

↳ Then by the rank-nullity Theorem, $\dim(\ker(T)) = 0$.

↳ hence $\ker(T) = \{\vec{0}\}$.

$\square$.