Lecture 19.

Ex: Let A be an $m\times n$ matrix and let
$$T_A : \mathbb{R}^n \to \mathbb{R}^m$$
be multiplication by A.

↳ if $m<n$, then T_A is not 1-1

↳ if $n<m$, then T_A is not onto.

↳ if $n=m$, then T_A is 1-1 and onto iff A is invertible i.e. iff $\det(A)\neq 0$.

From last time: If $\dim(V)=\dim(W)=n<\infty$, and $T:V\to W$ is linear, then TFAE:

1) T is 1-1
2) $\ker(T)=\{\vec{0}\}$.
3) T is onto.

Note that this does not hold in the infinite dimensional case: Let $V=\{(a_0,a_1,a_2,\dots) : a_i\in\mathbb{R}\ \forall i\}$ be the vector space of sequences. Then

1) $T_1 : V\to V$ given by
$$T_1((a_0,a_1,a_2,\dots)) = (0,a_0,a_1,a_2,\dots)$$

is 1-1 but not onto.

2) $T_2((a_0, a_1, a_2, \dots)) = (a_1, a_2, a_3, \dots)$

is onto, but not 1-1.

3) $T_3((a_0, a_1, a_2, \dots)) = (3a_0, 3a_1, 3a_2, \dots)$.

is 1-1 and onto.

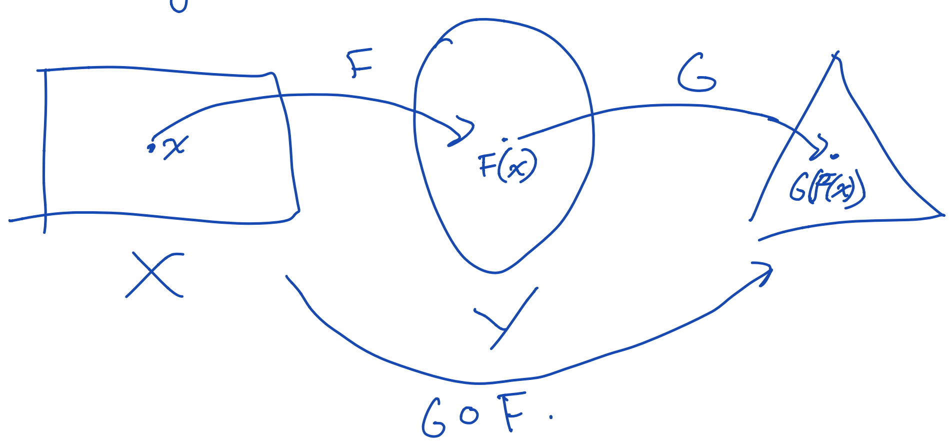
Composition of functions

Let $F: X \to Y$ and $G: Y \to Z$ be functions. The the composition of G with F

is the function

$$G \circ F : X \to Z$$

given by $G \circ F(x) = G(F(x))$.



<u>Theorem</u>: If $T_1: U \to V$ and $T_2: V \to W$ are linear transformations, then $T_2 \circ T_1 : U \to W$ is a linear transformation.

Pf: Check additivity, and homogeneity:

1) $T_2 \circ T_1(\vec{u}+\vec{v}) = T_2(T_1(\vec{u}+\vec{v}))$
$= T_2(T_1(\vec{u}) + T_1(\vec{v}))$
$= T_2(T_1(\vec{u})) + T_2(T_1(\vec{v})) = T_2\circ T_1(\vec{u}) + T_2\circ T_1(\vec{v})$.

2) $T_2 \circ T_1(k\vec{v}) = T_2(T_1(k\vec{v})) = T_2(kT_1(\vec{v})) = kT_2(T_1(\vec{v}))$
$= kT_2\circ T_1(\vec{v})$

$\square$

Ex: Let A be an $k\times n$ real matrix and let B be a $m\times k$ real matrix. Let $T_A : \mathbb{R}^n \to \mathbb{R}^k$ and $T_B : \mathbb{R}^k \to \mathbb{R}^m$. Then for $\vec{x}\in\mathbb{R}^n$,

$$T_B\circ T_A(\vec{x}) = T_B(A\vec{x}) = B(A\vec{x}) = (BA)\vec{x} = T_{BA}(\vec{x})$$

where BA is an $m\times n$ real matrix.

So $T_B\circ T_A = T_{BA}$.

Inverse Functions

Let $F : X \to Y$ be a function. Then F is <u>invertible</u> if there is a function $G : Y\to X$ such that $\forall x\in X$, $\forall y \in Y$,

$$F(x) = y \iff G(y) = x.$$

$G\circ F(x) = x$. We

Equivalently $F \circ G(y) = y$ [illegible]
say G is an inverse of F.

- If F is invertible, then it has a unique inverse $G = F^{-1}$.
- A function $F: X \to Y$ is invertible iff F is 1-1 and onto.

Theorem: Let $T: V \to W$ be a 1-1, onto linear transformation. Then $T^{-1}: W \to V$ is linear.

Pf: Let $\vec{w}_1, \vec{w}_2 \in W$ be arbitrary.

Then
$$\begin{aligned} T(T^{-1}(\vec{w}_1 + \vec{w}_2)) &= \vec{w}_1 + \vec{w}_2 \\ &= T(T^{-1}(\vec{w}_1)) + T(T^{-1}(\vec{w}_2)) \\ &= T(T^{-1}(\vec{w}_1) + T^{-1}(\vec{w}_2)). \end{aligned}$$

Since T is 1-1,
$$T^{-1}(\vec{w}_1 + \vec{w}_2) = T^{-1}(\vec{w}_1) + T^{-1}(\vec{w}_2) \checkmark.$$

Let k be a scalar and $\vec{w} \in W$ arbitrary.

Then
$$\begin{aligned} T(T^{-1}(k\vec{w})) = k\vec{w} &= kT(T^{-1}(\vec{w})) \\ &= T(kT^{-1}(\vec{w})). \end{aligned}$$

Since T is 1-1, $T^{-1}(k\vec{w}) = kT^{-1}(\vec{w})$.

$\square$.

Ex: Let A be an $n \times n$ complex matrix.
Let $T_A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be $T_A(\vec{x}) = A\vec{x}$.
Then T_A is invertible iff A is invertible, and
$T_A^{-1} = T_{A^{-1}}$. (Exercise).

Theorem: Let $T_1 : U \to V$ and $T_2 : V \to W$
be 1-1 and onto. Then

a) $T_2 \circ T_1 : U \to W$ is 1-1 and onto.

b) $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$

Pf: Exercise.