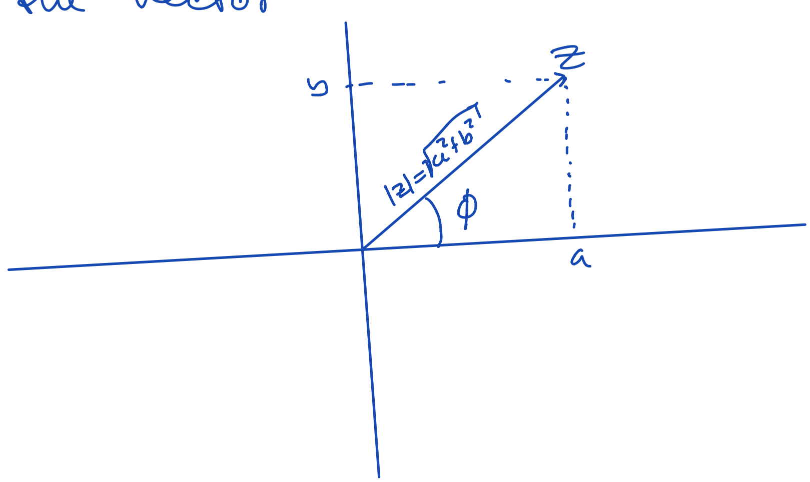
Lecture 2

Complex numbers, continued.

- Recall that a complex number is one of the form $z = a + bi$, where $a, b \in \mathbb{R}$ and $i^2 = -1$.

- the conjugate of $z = a+bi$ is $\bar{z} = a - bi$
- the modulus of $z = a+bi$ is $|z| = \sqrt{z\bar{z}} = \sqrt{a^2+b^2}$.
  - $\hookrightarrow$ the modulus is the length of the vector $(a,b) \in \mathbb{R}^2$ representing

- suppose $z = a+bi$. Then we can represent z as the vector



- If ϕ is the (counterclockwise) angle between the positive real axis then $a = |z| \cos\phi$ and $b = |z| \sin\phi$.

and so we may write

$$z = |z|(\cos\phi + i\sin\phi).$$

- this is the <u>polar form</u> of z.

- the angle ϕ is the argument.
- the argument is not unique, since

$$\left.\begin{aligned} \sin(\phi) &= \sin(\phi \pm 2\pi n) \\ \cos(\phi) &= \cos(\phi \pm 2\pi n) \end{aligned}\right. \quad \text{for any } n \in \mathbb{N}.$$

- we standardize and say ϕ is the <u>principle argument</u> if $-\pi \le \phi \le \pi$.

<u>Note:</u>

$$\sin\phi = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \phi^{2n+1}$$

$$\cos\phi = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \phi^{2n}.$$

so

$$\begin{aligned} z &= |z|(\cos\phi + i\sin\phi) \\ &= |z|\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \phi^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} i\phi^{2n+1}\right). \\ &= |z|\left(\left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \cdots\right) + \left(i\phi - \frac{i\phi^3}{3!} + \frac{i\phi^5}{5!} + \cdots\right)\right) \\ &= |z|\left(1 + \frac{(\phi i)^2}{2!} + \frac{(\phi i)^3}{3!} + \frac{(\phi i)^4}{4!} + \cdots\right) \end{aligned}$$

$= |z|\, e^{\phi i}$

$$\boxed{z = |z| e^{\phi i} = |z|(\cos\phi + i \sin\phi).}$$ very! nice.

multiplying and dividing in Polar form

Given z_1, z_2, say

$$z_1 = |z_1|(\cos\phi_1 + i\sin\phi_1) = |z_1| e^{i\phi_1}$$

$$z_2 = |z_2|(\cos\phi_2 + i\sin\phi_2) = |z_2| e^{i\phi_2}.$$

Then

$$\begin{aligned} z_1 z_2 &= |z_1||z_2| e^{i\phi_1} e^{i\phi_2} \\ &= |z_1 z_2|\, e^{i(\phi_1+\phi_2)} \\ &= |z_1 z_2| \left(\cos(\phi_1+\phi_2) + i\sin(\phi_1+\phi_2)\right). \end{aligned}$$

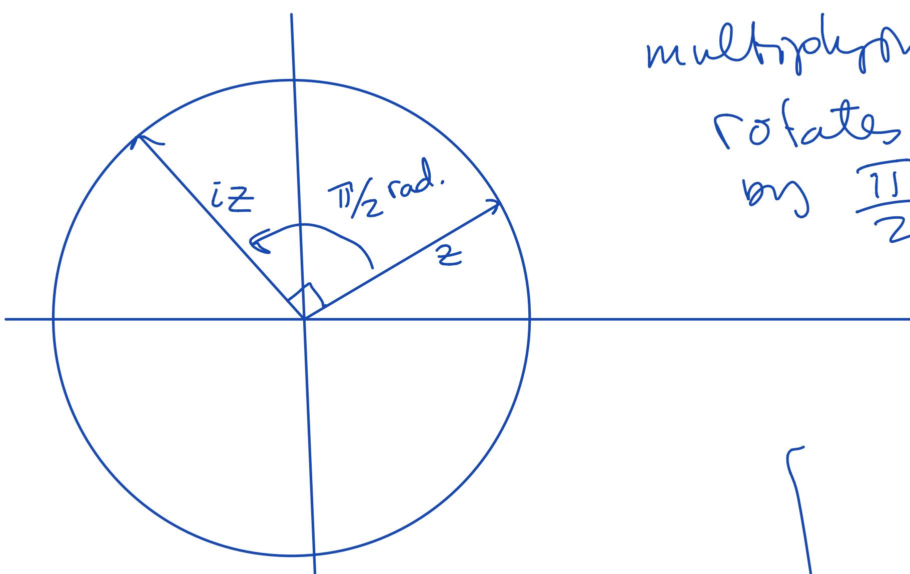
$$\begin{aligned} \frac{z_1}{z_2} &= \frac{|z_1|}{|z_2|} \frac{e^{i\phi_1}}{e^{i\phi_2}} \\ &= \frac{|z_1|}{|z_2|} e^{i(\phi_1-\phi_2)} \\ &= \frac{|z_1|}{|z_2|}\left(\cos(\phi_1-\phi_2) + i\sin(\phi_1-\phi_2)\right). \end{aligned}$$

Ex: $i = \underbrace{\cos\left(\frac{\pi}{2}\right)}_{=0} + i\,\underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1} = e^{\frac{i\pi}{2}}.$

Ex: $-1 = \underbrace{\cos(\pi)}_{-1} + i\,\underbrace{\sin(\pi)}_{0} = e^{i\pi}.$

$$e^{i\pi} + 1 = 0 \leftarrow \text{Euler's identity!}$$

Ex:
$$\underbrace{\begin{aligned} z^n &= \left(|z|e^{i\phi}\right)^n \\ &= |z|^n \left(e^{i\phi}\right)^n = |z^n|\, e^{in\phi} \\ &= |z^n|\left(\cos(n\phi) + i\sin(n\phi)\right). \end{aligned}}_{\text{De Moivre's identity.}}$$



multiplying by i rotates counterclockwise by $\frac{\pi}{2}$ radians.