Lecture 20

8.3 Isomorphisms.

Definition: A linear map $T: V \to W$ is called an **isomorphism** if T is 1-1 and onto.

Defn: Vector spaces V and W are said to be **isomorphic** (write $V \cong W$) if there exists an isomorphism from V to W (or equivalently from W to V ... why? Hint: if $T: V \to W$ is an isomorphism, so is $T^{-1}: W \to V$).

Isomorphisms are very important, since isomorphic vector spaces are "the same" algebraically.

Ex: consider the map

$$T: P^n(\mathbb{R}) \to \mathbb{R}^{n+1}$$

$$T(a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0) = (a_0, a_1, \dots, a_{n-1}, a_n).$$

$$T^{-1}: \mathbb{R}^{n+1} \to P^n(\mathbb{R})$$

$$T((a_0, a_1, \dots, a_n)) = a_n x^n + \dots + a_1 x + a_0.$$

Then

$$\begin{tikzcd}
(a_n x^n + \cdots + a_1 x + a_0) + (b_n x^n + \cdots + b_1 x + b_0) \arrow[r, "T"] \arrow[d, equal] & (a_0, \ldots, a_n) + (b_0, \ldots, b_n) \arrow[d, equal] \\
(a_n + b_n) x^n + \cdots + (a_1 + b_1) x + (a_0 + b_0) & (a_0 + b_0, \ldots, a_n + b_n) \arrow[l, "T^{-1}"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
k(a_n x^n + \cdots + a_1 x + a_0) \arrow[r, "T"] \arrow[d, equal] & k(a_0, a_1, \ldots, a_n) \arrow[d, equal] \\
k a_n x^n + \cdots + k a_1 x + k a_0 & (k a_0, k a_1, \ldots, k a_n) \arrow[l, "T^{-1}"]
\end{tikzcd}$$

So an isomorphism preserves the linear structure of a vector space; You can go back and forth. because they can be reversed i.e. they are invertible.

<u>Proposition</u>: Let V and W be K-vector spaces, and suppose $\dim(V) = n$. Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for V and let $\vec{w}_1, \ldots, \vec{w}_n$ be <u>any</u> n-vectors in W (not necessarily distinct). Then there is a <u>unique</u> linear map

$$T: V \to W$$

such that
$$T(\vec{v}_i) = \vec{w}_i \quad \text{for all } i.$$

Pf: For $\vec{v} \in V$, write $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$

and define

$$T(\vec{v}) = T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1\vec{w}_1 + \cdots + c_n\vec{w}_n.$$

Then T is linear, since if $\vec{u} = d_1\vec{v}_1 + \cdots + d_n\vec{v}_n$, then

$$\begin{aligned} T(\vec{v}+\vec{u}) &= T\big((c_1+d_1)\vec{v}_1 + \cdots + (c_n+d_n)\vec{v}_n\big) \\ &= (c_1+d_1)\vec{w}_1 + \cdots + (c_n+d_n)\vec{w}_n \\ &= (c_1\vec{w}_1 + \cdots + c_n\vec{w}_n) + (d_1\vec{w}_1 + \cdots d_n\vec{w}_n) \\ &= T(\vec{v}) + T(\vec{u}) \end{aligned}$$

and

$$\begin{aligned} T(k\vec{v}) &= T(kc_1\vec{v}_1 + \cdots + kc_n\vec{v}_n) \\ &= kc_1\vec{w}_1 + \cdots + kc_n\vec{w}_n = k(c_1\vec{w}_1 + \cdots + c_n\vec{w}_n) \\ &= kT(\vec{v}). \end{aligned}$$

Clearly $T(\vec{v}_i) = \vec{w}_i$ for all i.

Finally, T is unique: Let $S: V \to W$ be another linear map satisfying $S(\vec{v}_i) = \vec{w}_i$.

Let $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \in V$. Then

$$\begin{aligned} S(\vec{v}) = S(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) &= c_1 S(\vec{v}_1) + \cdots + c_n S(\vec{v}_n) \\ &= c_1\vec{w}_1 + \cdots + c_n\vec{w}_n \\ &= c_1 T(\vec{v}_1) + \cdots + c_n T(\vec{v}_n) \\ &= T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = T(\vec{v}). \end{aligned}$$

Hence $S = T$. $\square$.

Let $K = \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$.

Theorem: Let V and W be n-dimensional K-vector spaces. Then $V \cong W$.

Pf: Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis for V and $\{\vec{w}_1, \dots, \vec{w}_n\}$ be a basis for W. By the previous proposition, there is a unique linear map $T: V \to W$ such that $T(\vec{v}_i) = \vec{w}_i \ \forall i$.

We need to check that T is 1-1 and onto.

1-1: Let $\vec{v} \in \ker(T)$. WTS: $\vec{v} = 0$. Since $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis, we can write

$$\vec{v} = c_1 v_1 + \cdots + c_n \vec{v}_n.$$

Since $\vec{v} \in \ker(T)$,

$$\vec{0} = T(\vec{v}) = c_1 T(\vec{v}_1) + \cdots + c_n T(\vec{v}_n)$$
$$= c_1 \vec{w}_1 + \cdots + c_n \vec{w}_n.$$

Since $\{\vec{w}_1, \dots, \vec{w}_n\}$ is a basis for W, $c_1 = \cdots = c_n = 0$.

Hence $\vec{v} = 0$. ✓

Onto: Let $\vec{w} \in W$ be arbitrary. We want to find $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$.

Since $\{\vec{w}_1, \dots, \vec{w}_n\}$ is a basis for W, there are unique $c_1, \dots, c_n \in K$ such that

$$\vec{w} = c_1\vec{w}_1 + \cdots + c_n\vec{w}_n.$$

Let $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \in V$ (same scalars!).

Then
$$\begin{aligned} T(\vec{v}) &= T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) \\ &= c_1 T(\vec{v}_1) + \cdots + c_n T(\vec{v}_n) \\ &= c_1\vec{w}_1 + \cdots + c_n\vec{w}_n = \vec{w}. \checkmark \end{aligned}$$

Hence $T: V \to W$ is an isomorphism and so $V \cong W$. $\square$.

<u>Corollary:</u> Let $K = \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$. Let V be a K-vector space with $\dim(V) = n < \infty$. Then $V \cong K^n$.

<u>Pf:</u> Apply the previous theorem. $\square$.

<u>Notation:</u> If $V \cong K^n$, $\vec{v} \in V$, and β is a basis for K^n, then $[\vec{v}]_B$ is the matrix/vector of coordinates of $\vec{v}$ with respect to B. The map
$$\begin{aligned} V &\to K^n \\ \vec{v} &\mapsto [\vec{v}]_B \end{aligned}$$
is an <u>isomorphism</u>.

As an application, we can represent linear maps of finite dimensional vector spaces.

Ex: Let $T : P_1(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $T(p(x)) = x\,p(x)$.

We have that $P_1(\mathbb{R}) \cong \mathbb{R}^2$ and $P_2(\mathbb{R}) \cong \mathbb{R}^3$

So

$$
\begin{tikzcd}
\vec{v} = a_0 + a_1 x \arrow[d, leftrightarrow] & \in P_1(\mathbb{R}) \arrow[rr, "T"] \arrow[d, leftrightarrow, "\cong"'] & & P_2(\mathbb{R}) \arrow[d, leftrightarrow, "\cong"'] & \ni T(a_0 + x) = a_0 x + a_1 x^2 \arrow[d, leftrightarrow] \\
{[\vec{v}]}_{\{\vec{e}_1, \vec{e}_2\}} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} & \in \mathbb{R}^2 \arrow[rr, "\underbrace{T_A}_{\text{for some matrix } A.}"'] & & \mathbb{R}^3 & \begin{bmatrix} 0 \\ a_0 \\ a_1 \end{bmatrix}
\end{tikzcd}
$$

$$A = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix}, \quad A \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ a_0 \\ a_1 \end{bmatrix}$$

So we get

$$x_1 a_0 + x_2 a_1 = 0.$$
$$y_1 a_0 + y_2 a_1 = a_0.$$
$$z_2 a_0 + z_2 a_1 = a_1.$$

for any a_0, a_1. Hence $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.