Lecture 22

8.5 Change of Basis and Similarity.

Change of Basis

- Let $K = \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$.
- Let V be a K-vector space, $\dim(V)=n<\infty$, with two bases, $B=\{\vec{u}_1,\dots,\vec{u}_n\}$ and $B'=\{\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n\}$.
- Question: How are B and B' related?
- Consider the identity operator $I: V\to V$ defined by $I(\vec{v})=\vec{v}$ for all $\vec{v}\in V$.
- From the last lecture we have, for any $\vec{v}\in V$

$$\begin{tikzcd}
\vec{v} \arrow[r, "I"] \arrow[d] & \vec{v} \arrow[d] \\
K^n \ni [\vec{v}]_B \arrow[r, "{[I]_{B',B}}"] & {[\vec{v}]_{B'} \in K^n}
\end{tikzcd}$$

Definition: $P_{B\to B'} := [I]_{B',B}$ is called the transition matrix from B to B'.

Ex: In $\mathbb{R}^2$, the sets

$$B = \underbrace{\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}}_{\text{standard basis}} \text{ and } B' = \left\{ \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \end{bmatrix} \right\}$$

are bases for $\mathbb{R}^2$. Then, by the above

$$I\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 4 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

↳ so $\left[I\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \right]_{B'} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

$$I\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -7\begin{bmatrix} 4 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

↳ so $\left[I\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \right]_{B'} = \begin{bmatrix} -7 \\ 4 \end{bmatrix}$.

Hence $P_{B \to B'} = \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$

Note: For any $\vec{v} \in \mathbb{R}^n$

$$P_{B \to B'} [\vec{v}]_B = [\vec{v}]_{B'}.$$

For example, let B, B' be as in the last example. What is $\begin{bmatrix}2\\3\end{bmatrix}_{B'}$?

$\begin{bmatrix}2\\3\end{bmatrix}_B = \begin{bmatrix}2\\3\end{bmatrix}$, since $\begin{bmatrix}2\\3\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix}$

Then $\begin{bmatrix}2\\3\end{bmatrix}_{B'} = \begin{bmatrix}2 & -7\\-1 & 4\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}-17\\10\end{bmatrix}$.

And indeed
$$-17\begin{bmatrix}4\\1\end{bmatrix} + 10\begin{bmatrix}7\\2\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix} \checkmark.$$

- Of course, we can go "backwards"

$$P_{B'\to B} = [I]_{B,B'} = \left[[\vec{v}_1]_B \Big| \cdots \Big| [\vec{v}_n]_B\right].$$

Ex: $B' = \left\{\begin{bmatrix}4\\1\end{bmatrix}, \begin{bmatrix}7\\2\end{bmatrix}\right\}$, $B = \{\vec{e}_1, \vec{e}_2\}$.

Then $P_{B'\to B} = \left[\begin{bmatrix}4\\1\end{bmatrix}_B \; \begin{bmatrix}7\\2\end{bmatrix}_B\right] = \begin{bmatrix}4 & 7\\1 & 2\end{bmatrix}$.

Note that: $\begin{bmatrix}4 & 7\\1 & 2\end{bmatrix}\begin{bmatrix}2 & -7\\-1 & 4\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

This is to be expected, since, as I is invertible and so from last time, $P_{B\to B'}$ is invertible and

$$\left(P_{B\to B'}\right)^{-1} = P_{B'\to B}.$$

Similarity:

Recall that matrices A and B are similar if there is an invertible matrix P such that

$$B = PAP^{-1}.$$

Similar matrices arise in the following question:

Question: Let $T: V \to V$ be a linear operator. Let B and B' be bases for V. How are $[T]_B$ and $[T]_{B'}$ related?

Answer:

$$\begin{tikzcd}
V \ni \vec{v} \arrow[r, "T"] \arrow[d] & T(\vec{v}) \in V \arrow[d] & V \ni \vec{v} \arrow[r, "T"] \arrow[d] \arrow[dll, bend left, green] & T(\vec{v}) \in V \arrow[d] \\
\mathbb{R}^n \ni [\vec{v}]_B \arrow[r, "{[T]_B}"] & {[T(\vec{v})]_B} \in \mathbb{R}^n \arrow[rr, bend right, red] & {[\vec{v}]_{B'}} \arrow[r, "{[T]_{B'}}"] & {[T(\vec{v})]_{B'}}
\end{tikzcd}$$

$P_{B' \to B} = (P_{B \to B'})^{-1}$ $\qquad$ $P_{B \to B'}$

- So "chasing the diagram" see that for $\vec{v} \in V$,

$$[T]_{B'} [\vec{v}]_{B'} = P_{B \to B'} [T]_B \, P_{B' \to B} [\vec{v}]_{B'}$$
$$= P_{B \to B'} [T]_B (P_{B \to B'})^{-1} [\vec{v}]_{B'}$$

- That is, $[T]_{B'}$ and $[T]_B$ are similar matrices!

- In fact we have the following theorem:

Theorem: Let V be a vector space with $\dim(V) = n$. Then two $n \times n$ matrices C and D are similar if and only if $C = [T]_B$ and $D = [T]_{B'}$ for some linear operator $T: V \to V$ and bases B and B' of V. Furthermore if $D = PCP^{-1}$. Then $P = P_{B \to B'}$

Significance:

- If C and D are matrices, then:
  - C and D have the same determinant, characteristic poly, eigenvalues, and dimension of eigenspaces.

- Therefore given a linear operator $T: V \to V$ and a basis B of V, we can define:
  - $\det(T) = \det([T]_B)$
  - λ is an eigenvalue of T iff λ is an eigenvalue of $[T]_B$.
  - the eigenspace of λ is $\ker(\lambda I - T)$.

- More generally, if $T: V \to V$ is a linear operator, then λ is an eigenvalue if there exists $\vec{v} \in V$ such that $T(\vec{v}) = \lambda \vec{v}$. The vector $\vec{v}$ is an eigenvector of λ.

Ex: Let $V = C^{\infty}(-\infty, \infty)$, the set of functions with continuous derivatives of any order (i.e. the smooth functions).

Then let $D: V \to V$ be $D(f) = f'$.

Then for any λ, if $f = e^{\lambda x}$, then

$$D(f) = \lambda e^{\lambda x},$$ so λ is an eigenvalue of D with eigenvector $e^{\lambda x}$.