Lecture 23

Starting Chapter 7.

7.1 Orthogonal Matrices

Definition: Let A be an $n\times n$ REAL (or rational) matrix. We say that A is orthogonal iff

$$A^T = A^{-1},$$

equivalently $$AA^T = A^TA = I$$

- "orthogonal" is a bit of a misnomer, since

Theorem: Let A be an $n\times n$ real matrix. TFAE.

1) A is an orthogonal matrix
2) The columns of A are orthonormal with respect to the dot product on $\mathbb{R}^n$.
3) The rows of A are orthonormal with respect to the dot product on $\mathbb{R}^n$

Proof is straightforward. Exercise.

Examples: The rotation matrices.

Let θ be an angle. Then

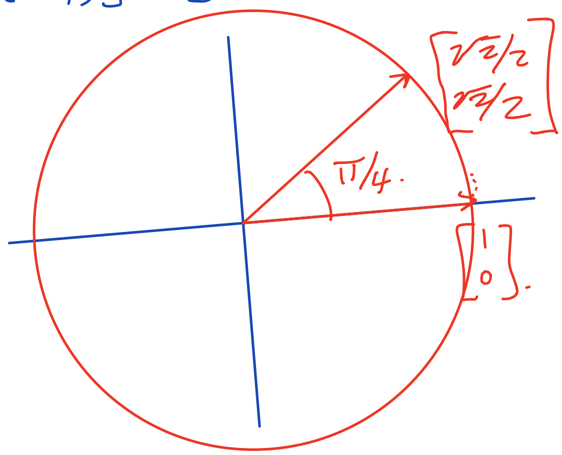
$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is a rotation matrix. Applying A to a vector $\vec{v} \in \mathbb{R}^2$ rotates $\vec{v}$ θ radians counterclockwise.

Example if $\theta = \frac{\pi}{4}$. Then

$$A = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$



A is always orthogonal

$$AA^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \cos\theta\sin\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Theorem:

1) The transpose of an orthogonal matrix is orthogonal
2) The inverse of an orthogonal matrix is orthogonal.
3) The product of orthogonal matrices is orthogonal.
4) If A is orthogonal, then $\det(A) = \pm 1$.

- This theorem essentially says that forms a special sort of structure called a "group"!
- The set of $n \times n$ orthogonal matrices is called the orthogonal group, $O(n)$.

The proofs are also easy. Ex:

4) Let A be orthogonal. Then

$$1 = \det(I) = \det(AA^T) = \det(A)\det(A^T) = \det(A)^2$$

so $\det(A) = \pm\sqrt{1} = \pm 1$. $\square$

- It turns out that the orthogonal matrices are precisely the ones that transform vectors without changing their length.

Theorem: Let A be a real $n \times n$ matrix. TFAE:

1) A is orthogonal.
2) $\|A\vec{v}\| = \|\vec{v}\| \quad \forall \vec{v} \in \mathbb{R}^n$.
3) $(A\vec{v}) \cdot (A\vec{w}) = \vec{v} \cdot \vec{w} \quad \forall \vec{v}, \vec{w} \in \mathbb{R}^n$.

Pf: We show 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 1)

1) $\Rightarrow$ 2): (Recall that $\vec{v} \cdot \vec{w} = \vec{v}^T \vec{w}$).

Then $\|A\vec{v}\|^2 = (A\vec{v}) \cdot (A\vec{v})$

$$= (A\vec{v})^T (A\vec{v})$$
$$= \vec{v}^T A^T A \vec{v} = \vec{v}^T I \vec{v} = \vec{v}^T \vec{v} = \|\vec{v}\|^2.$$

2) $\Rightarrow$ 3) We need to use an identity:

$$\vec{v} \cdot \vec{w} = \frac{1}{4}\|\vec{v} + \vec{w}\| - \frac{1}{4}\|\vec{v} - \vec{w}\| \quad (*)$$

Then

$$(A\vec{v})\cdot(A\vec{w}) = \frac{1}{4}\|A\vec{v} + A\vec{w}\| - \frac{1}{4}\|A\vec{v} - A\vec{w}\|$$
$$= \frac{1}{4}\|A(\vec{v}+\vec{w})\| - \frac{1}{4}\|A(\vec{v}-\vec{w})\|$$
$$= \frac{1}{4}\|\vec{v}+\vec{w}\| - \frac{1}{4}\|\vec{v}-\vec{w}\| \quad \text{(by 2))}$$
$$= \vec{v}\cdot\vec{w} \quad \text{(by } (*)\text{)}.$$

3) $\Rightarrow$ 1): By 3) for all vectors $\vec{v}$ and $\vec{w} \in \mathbb{R}^n$

$$\vec{v}^T A^T A \vec{w} = \vec{v}^T \vec{w} = \vec{v}^T I_n \vec{w}$$

Hence $\vec{v}^T A^T A \vec{w} - \vec{v}^T I_n \vec{w} = \vec{0}$

so $\vec{v}^T (A^T A - I_n)\vec{w} = \vec{0}$.

In particular, $\vec{e}_i^T (A^T A - I_n)\vec{e}_j = \vec{0} \quad \forall i, j \leq n.$

Hence $I_n^T (A^T A - I_n) I_n = [0]$ ← the zero matrix.

and so $A^T A - I_n = [0]$

i.e. $A^T A = I_n$. So A is orthogonal.

$\square$.

Conclusion: The orthogonal matrices are precisely the matrices which preserve distance wrt the Euclidean norm and which preserve angles between vectors:

$$\arccos\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|}\right) = \arccos\left(\frac{A\vec{v} \cdot A\vec{w}}{\|A\vec{v}\| \|A\vec{w}\|}\right).$$