Lecture 24

Recall that a real matrix A is orthogonal if $AA^T = A^TA = I$.

Equivalently A is orthogonal if the columns of A are orthonormal.

Theorem: Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional real innerproduct space and let $B = \{\vec{b}_1, \dots, \vec{b}_n\}$ and $B' = \{\vec{b}_1', \dots, \vec{b}_n'\}$ be orthonormal bases for V. Then $P_{B \to B'}$ is orthogonal.

Pf: Recall that

$$P_{B\to B'} = \left[[\vec{b}_1]_{B'} \mid \cdots \mid [\vec{b}_n]_{B'} \right].$$

Since B' is orthonormal,

$$\vec{b}_i = \langle \vec{b}_i, \vec{b}_1' \rangle \vec{b}_1' + \langle \vec{b}_i, \vec{b}_2' \rangle \vec{b}_2' + \dots + \langle b_i, \vec{b}_n' \rangle \vec{b}_n.$$

and so

$$[\vec{b}_i]_{B'} = \begin{bmatrix} \langle \vec{b}_i, \vec{b}_1' \rangle \\ \langle \vec{b}_i, \vec{b}_2' \rangle \\ \vdots \\ \langle \vec{b}_i, \vec{b}_n' \rangle \end{bmatrix}$$

Hence $P_{B \to B'} = \begin{bmatrix} \langle \vec{b}_1, \vec{b}_1' \rangle & \langle \vec{b}_2, \vec{b}_1' \rangle & \cdots & \langle \vec{b}_n, \vec{b}_1' \rangle \\ \langle \vec{b}_1, \vec{b}_2' \rangle & \langle \vec{b}_2, \vec{b}_2' \rangle & \cdots & \langle \vec{b}_n, \vec{b}_2' \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \vec{b}_1, \vec{b}_n' \rangle & \langle \vec{b}_2, \vec{b}_n' \rangle & \cdots & \langle \vec{b}_n, \vec{b}_n' \rangle \end{bmatrix}$

By the same reasoning,

$$P_{B' \to B} = \begin{bmatrix} \langle \vec{b}_1', \vec{b}_1 \rangle & \langle \vec{b}_2', \vec{b}_1 \rangle & \cdots & \langle \vec{b}_n', \vec{b}_1 \rangle \\ \langle \vec{b}_1', \vec{b}_2 \rangle & \langle \vec{b}_2', b_2 \rangle & & \langle \vec{b}_n', \vec{b}_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \vec{b}_1', \vec{b}_n \rangle & \langle \vec{b}_2', \vec{b}_n \rangle & & \langle \vec{b}_n', \vec{b}_n \rangle \end{bmatrix}$$

$$= \left(P_{B \to B'}\right)^T.$$

But also, $P_{B' \to B} = \left(P_{B \to B'}\right)^{-1}$,

Hence $\left(P_{B \to B'}\right)^T = \left(P_{B - B'}\right)^{-1}$.

So $P_{B \to B'}$ is orthogonal. $\square$.

7.2 Orthogonal Diagonalization

Recall that A is diagonalizable if there is an invertible matrix P and a Diagonal matrix D such that $A = P^{-1}DP$.

- In general, it is difficult to determine if a matrix is diagonalizable.

- Today, we aim to show that all real symmetric matrices are diagonalizable in a nice way.

Defn: Let A and B be square real matrices. We say that A and B are <u>orthogonally similar</u> if there is an orthogonal matrix P such that $B = P^{-1}AP$.

If A is orthogonally similar to a diagonal matrix D, then we say A is orthogonally diagonalizable

Theorem: Let A be a real $n \times n$ matrix. TFAE:

a) A is orthogonally diagonalizable

b) A has an orthonormal set of n eigenvectors (i.e. $\mathbb{R}^n$ has an orthonormal basis of eigenvectors of A).

c) A is symmetric.

Pf: a) $\Rightarrow$ b).

· Assume that $P^T A P = \begin{bmatrix} \lambda_1 & \cdots & & 0 \\ \vdots & \lambda_2 & & \vdots \\ & & \ddots & \\ 0 & \cdots & & \lambda_n \end{bmatrix}$

- Write $P = [\vec{v}_1 | \dots | \vec{v}_n]$.
- Then $AP = PD \iff A\vec{v}_j = \lambda_j \vec{v}_j$, $1 \le j \le n$.
- Hence the $\vec{v}_i$ are eigenvectors of A, and since they are columns of P are orthonormal, A has an orthonormal set of eigenvectors.

b) ⇒ c). Suppose A is orthogonally diagonalizable. Then $A = PDP^T$ for orthogonal matrix P and diagonal matrix D.

· Then $A^T = (PDP^T)^T = (P^T)^T (PD)^T$
$= PD^T P^T = PDP^T$ (since D is symmetric)

· Hence $A^T = A$. ✓.

c) ⇒ a) Hard. See exercise 31, chapter 7.2.

$\boxed{\cdot}$

b) ⇒ a): See section on diagonalizability.

Theorem: Let A be a real, symmetric matrix.
Then
a) The eigenvalues of A are real.
b) eigenvectors corresponding to distinct eigenvalues are orthogonal.

Pf:
a) Let $\lambda \in \mathbb{C}$ be an eigenvalue of A.
W.T.S.: $\lambda \in \mathbb{R}$.

- Let $\vec{v} \in \mathbb{C}^n$ be a non-zero eigenvector of A.
- Let $\langle \cdot , \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be the complex Euclidean inner product.
- Compute $\langle A\vec{v}, \vec{v} \rangle$ in two ways.

1) $\langle A\vec{v}, \vec{v} \rangle = \langle \lambda\vec{v}, \vec{v} \rangle = \lambda \langle \vec{v}, \vec{v} \rangle$.

2) $\langle A\vec{v}, \vec{v} \rangle = \langle \vec{v}, \overline{A^T}\vec{v} \rangle = \langle \vec{v}, A\vec{v} \rangle$
$= \langle \vec{v}, \lambda\vec{v} \rangle$
$= \bar{\lambda} \langle \vec{v}, \vec{v} \rangle$.

Since $\langle \vec{v}, \vec{v} \rangle \neq 0$, $\lambda = \bar{\lambda}$. Hence $\lambda \in \mathbb{R}$.

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b). Let $\lambda \neq \mu$ be eigenvalues of A.
Let $A\vec{v} = \lambda\vec{v}$ and $A\vec{w} = \mu\vec{w}$, $\vec{v} \neq 0, \vec{w} \neq 0$.

$(A\vec{v})\cdot(\vec{w}) = (\lambda\vec{v})\cdot w = \lambda(\vec{v}\cdot\vec{w}).$

But also

$$
\begin{aligned}
A\vec{v}\cdot\vec{w} = (A\vec{v})^T\vec{w} &= \vec{v}^T A^T\vec{w} \\
&= \vec{v}\cdot(A^T\vec{w}) \\
&= \vec{v}\cdot(A\vec{w}) \quad \text{(by symmetry)} \\
&= \vec{v}\cdot\mu\vec{w} \\
&= \mu(\vec{v}\cdot\vec{w}).
\end{aligned}
$$

Hence
$$
\begin{aligned}
0 &= \lambda(\vec{v}\cdot\vec{w}) - \mu(\vec{v}\cdot\vec{w}) \\
&= (\lambda-\mu)(\vec{v}\cdot\vec{w}).
\end{aligned}
$$

Since $\lambda - \mu \neq 0$ it must be that $\vec{v}\cdot\vec{w} = 0$.

$\square$