Lecture #3.

4.1 Vector spaces.

- Here we present the formal definition of a vector space.
- Let K be $\mathbb{C}$ or $\mathbb{R}$ (or $\mathbb{Q}$)

Defn A set V together with operations

$$+ : V \times V \to V \quad \text{(addition)}$$
$$(w, v) \mapsto \vec{w} + \vec{v}$$
$$\cdot : K \times V \to V \quad \text{scalar mult.}$$
$$(c, \vec{v}) \mapsto c\vec{v}$$

is called a K-vector space iff:

1) $\forall \vec{v}, \vec{w} \in V$, $\quad \vec{v} + \vec{w} \in V$
2) (Associativity of $+$) $\forall \vec{u}, \vec{v}, \vec{w}$, $\quad \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
3) (Commutativity of $+$) $\forall \vec{v}, \vec{w} \in V$ $\quad \vec{v} + \vec{w} = \vec{w} + \vec{v}$.
4) There is an object $\vec{0} \in V$ such that $\forall \vec{v} \in V$ $\quad \vec{0} + \vec{v} = \vec{v}$.
5) For each $\vec{v} \in V$, there is $-\vec{v} \in V$ such that $\vec{v} + (-\vec{v}) = \vec{0}$.
6) (Scalar multiplication) For every $c \in K$ and every $\vec{v} \in V$, $\quad c\vec{v} \in V$.

7) For every $c \in K$, $c(\vec{u}+\vec{v}) = c\vec{u} + c\vec{v}$

8) For every $c, d \in K$, $(c+d)\vec{v} = c\vec{v} + d\vec{v}$

9) $c(d\vec{v}) = (cd)\vec{v}$

10) $1\vec{v} = \vec{v}$

<u>Note:</u> Many of the axioms are immediate from the fact that $K = \mathbb{C}$ or $\mathbb{R}$.

For now, we focus on $\mathbb{R}$-vectorspaces and then study $\mathbb{C}$-vectorspaces next week.

<u>Remark:</u> Given a set V with two operations, the most important axioms to check are 1, 6. The others usually follow easily.

Examples:

1) The zero vector space: $k = \mathbb{R}$ or $\mathbb{C}$ or $\mathbb{Q}$.

$V = \{\vec{0}\}$ ← the set consists of a single element.

- Addition: $\vec{0} + \vec{0} = \vec{0}$.
- scalar multiplication: $c\,\vec{0} = \vec{0}$ for all $c \in k$.

2) $\mathbb{R}^n$

$$V = \{(a_1, \dots, a_n) : a_1, \dots, a_n \in \mathbb{R}\}.$$

- Addition:

$$(a_1, \dots, a_n) + (b_1, \dots, b_n) = (a_1 + b_1, \dots, a_n + b_n)$$

- Scalar mult:

$$c(a_1, \dots, a_n) = (ca_1, \dots, ca_n).$$

3) $\mathbb{R}^\infty$ - infinite sequences of real #s.

$$V = \{(a_1, a_2, \dots, a_n, \dots), \ a_i \in \mathbb{R}\}.$$

Addition: $(a_1, a_2 \dots) + (b_1, b_2, \dots)$
$= (a_1 + b_1, \cancel{a}a_2 + b_2, \dots)$.

scalar mult: $c(a_1, a_2, \dots) = (ca_1, ca_2, \dots)$.

4): The set $M_{m\times n}(\mathbb{R})$ of $m\times n$ real matrices:

$$V = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} : a_{ij} \in \mathbb{R},\ 1 \le i \le m,\ 1 \le j \le n \right\}$$

<u>Addition:</u> matrix addition: for $A, B \in V$,

$$(A+B)_{ij} = (A)_{ij} + (B)_{ij}$$

<u>Scalar mult:</u> For $A \in V$,

$$(cA)_{ij} = c(A)_{ij}$$

See examples 6, 7, 8 from text.

<u>Theorem:</u> Let V be a V.S., $\vec{v} \in V$, c a scalar.

Then:

a) $0\vec{v} = \vec{0}$

b) $c\vec{0} = \vec{0}$

c) $(-1)\vec{v} = -\vec{v}$

d) If $c\vec{v} = \vec{0}$ then $c = 0$ or $\vec{v} = \vec{0}$.

<u>Pf:</u> a) $0\vec{v} + 0\vec{v} = (0+0)\vec{v}$ (by axiom 8)

$= 0\vec{v}.$

By axiom 5) there exists a negative $-0\vec{v}$ of $0\vec{v}$ such that $0\vec{v}-0\vec{v}=\vec{0}$. Hence

$$(0\vec{v}+0\vec{v})-0\vec{v} = 0\vec{v}-0\vec{v}$$

By axiom 3:

$$0\vec{v}+(0\vec{v}-0\vec{v}) = 0\vec{v}-0\vec{v}$$

so $$0\vec{v}+\vec{0} = \vec{0} \qquad \text{(by Axiom 5)}$$

so $$0\vec{v} = \vec{0} \qquad \text{by axiom 4).}$$

b) Let $\vec{v} \in V$ be an arbitrary vector.

By axiom 8)

$$c\vec{0}+c\vec{v} = c(\vec{0}+\vec{v})$$

By axiom 4) $$c\vec{0}+c\vec{v} = c\vec{v}.$$

By 5), $c\vec{v}$ has a negative $-c\vec{v}$, and so

$$(c\vec{0}+c\vec{v})-c\vec{v} = c\vec{v}-c\vec{v} = \vec{0}$$

By axiom 3) $$c\vec{0}+(c\vec{v}-c\vec{v}) = \vec{0}$$

By 5) $$c\vec{0}+\vec{0} = \vec{0}$$

By 4) $$c\vec{0} = \vec{0}.$$

c) To show that $(-1)\vec{v} = -\vec{v}$, we show $\vec{v} + (-1)\vec{v} = \vec{0}$.

$$\vec{v} + (-1)\vec{v} = 1\vec{v} + (-1)\vec{v} \quad \text{(by 10)}$$
$$= (1 + (-1))\vec{v} \quad \text{(by 8)}$$
$$= 0\vec{v}$$
$$= \vec{0} \quad \text{by (a).}$$

d) Suppose $c\vec{v} = \vec{0}$. we break the proof into cases:

i) Assume $c \neq 0$ (then show $\vec{v} = \vec{0}$).

Since $c \neq 0$, we have

$$c\vec{v} = \vec{0}$$

so $\frac{1}{c}(c\vec{v}) = \frac{1}{c}\vec{0}$

so $1\vec{v} = \frac{1}{c}\vec{0}$ (by 9).

$$\vec{v} = \frac{1}{c}\vec{0} \quad \text{(by 10)}$$
$$\vec{v} = \vec{0} \quad \text{by part (b).}$$

ii) Assume $\vec{v} \neq 0$ (then show $c = 0$).

Since $\vec{v} \neq 0$, there is $\vec{u} \in V$ such that $\vec{u} + \vec{v} \neq \vec{u}$ (by axiom 4).

For a contradiction, assume $c \neq 0$.

Then $c(\vec{u} + \vec{v}) \neq c\vec{u}$

so $c\vec{u} + c\vec{v} \neq c\vec{u}$ by axiom 2.

so $c\vec{u} + \vec{0} \neq c\vec{u}$ (by assumption).

so $c\vec{u} \neq c\vec{u}$, a contradiction.