Lecture #4.

4.2 (Subspaces).

Let V be a vector space.

<u>Defn</u>: A subset $W \subseteq V$ is a <u>subspace</u> of V iff W is also a vector space under the <u>same</u> addition and scalar multiplication operation on W.

- If V is a vector space, and $W \subseteq V$ a subset, then W "inherits" some of the properties of satisfied by V.

- To see that $W \subseteq V$ is a subspace, we only need to check the following axioms:

  Axiom 1: Closure under addition.
  Axiom 4: Existence of a zero vector.
  Axiom 5: Existence of a negative in W for every vector w.
  Axiom 6: Closure under scalar multiplication.

However, we don't even need to check all of these:

<u>Theorem:</u> Let $W \subseteq V$. Then W is a subspace iff the following axioms hold!

a) W is closed under addition (Axiom 1)

b) W is closed under scalar mult. (Axiom 6)

Pf: If W is a subspace then it is itself a vector space. Therefore it satisfies axiom 1, and axiom 6.

On the other hand, suppose W satisfies Axiom 1 and 6. Then we need only to show that W satisfies Axiom 4 and 5. To see this, note that for $\vec{w} \in W$,

$$0 \cdot \vec{w} = \vec{0} \in W$$

and

$$(-1)\,\vec{w} = -\vec{w},$$

so we are done. $\square$.

Examples:

- The zero vector space is a subspace of every vectorspace.
- A line through the origin is a subspace of $\mathbb{R}^2$ or $\mathbb{R}^3$.
- A plane through the origin is a subspace of $\mathbb{R}^3$...

Theorem: If $W_1, \ldots W_k$ are subspaces of V. Then $\bigcap_{i=1}^{k} W_i$ is a subspace of V.

Another way to get subspaces: spanning sets.

If V is a vector space, and $S = \{\vec{v}_1, \ldots, \vec{v}_k\} \subseteq V$, then $\text{span}(S) = \{ \sum_{i=1}^{k} c_i \vec{v}_i : c_i \in K \}$ is the smallest subspace of V containing S.

- We say that $\text{Span}(S)$ is the subset generated by S.

<u>Ex:</u> In $\mathbb{R}^3$, let $\vec{e}_1 = (1,0,0)$ and $\vec{e}_2 = (0,1,0)$.
Then $\text{span}(\vec{e}_1, \vec{e}_2)$ is the xy-plane.

<u>Kernels:</u>

<u>Theorem:</u> Let A be an $m \times n$ matrix with associated linear map $T_A : \mathbb{R}^n \to \mathbb{R}^m$. Then $\ker(T_A) = \{ \vec{v} \in \mathbb{R}^n : T_A(\vec{v}) = \vec{0} \}$ is a subspace.

In fact, this is true for <u>any</u> linear map $T: V \to W$. $\ker(T)$ is a subspace of V (EXERCISE!).

<u>Note:</u> Spanning sets are not unique!!

<u>4.3 Linear Independence.</u>

- Let V be a vector space

<u>Defn:</u> A set $\{ \vec{v}_1, \ldots, \vec{v}_n \} \subseteq V$ is called linearly independent iff no vector in the set can be written as a linear combination of the others.

- In practice the definition is not a very efficient way of checking linear independence.

- The following is usually easier:

Theorem: A set $\{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly independent iff the only coefficients satisfying

$$k_1\vec{v}_1 + \dots + k_n\vec{v} = \vec{0}$$

are $k_1 = k_2 = \dots = k_n = 0$.

Another way to phrase this: write $A = [\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n]$. Then $A\vec{x} = \vec{0}$ has only $\bar{x} = 0$ as a solution.

Ex: Determine if

$\vec{v}_1 = (1, -2, 3)$ $\vec{v}_2 = (5, 6, -1)$, $\vec{v}_3 = (3, 2, 1)$.

are linearly independent.

Soln: We form the matrix

$$A = [\vec{v}_1 \ \ \vec{v}_2 \ \ \vec{v}_3] = \begin{bmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{bmatrix}$$

We want a solution to the equation

$$A\vec{x} = \vec{0}$$

By Gauss-Jordan A is equivalent to

$$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

and so $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1/2\, t \\ -1/2\, t \\ t \end{pmatrix}$ is the general solution.

$t \neq 0$ gives a non-trivial solution, so $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are not independent.

Ex: The set of polynomials of degree n is a vector space.

$\{1, x, x^2, \ldots, x^n\}$ is a spanning set.

Theorem: a) A finite set that contains $\vec{0}$ is linearly dependent.

b) A set with one vector $\{\vec{v}\}$ is indep iff $\vec{v} \neq \vec{0}$.

c) $\{\vec{v}_1, \vec{v}_2\}$ is indep iff $\vec{v}_1 \neq c\vec{v}_2$.