Lecture #5

4.4 Coordinates and Basis

- a line is 1-dimensional
- a plane is 2-dim.
- the room is three dimensional.

Definition: Let V be a vector space. we say that V is finite dimensional if there is a finite set $\{\vec{v}_1, \dots, \vec{v}_n\} \subseteq V$ such that $\text{span}(\vec{v}_1, \dots, \vec{v}_n) = V$.

If V is not finite dimensional, we say that V is infinite dimensional.

Definition: Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\} \subseteq V$, a finite dimensional vector space. Then $\mathcal{B}$ is a **basis** of V iff

a) $\text{Span}(\mathcal{B}) = V$.

b) $\mathcal{B}$ is linearly independent.

Ex: (standard basis for K^n, $K = \mathbb{R}$ or $\mathbb{C}$)

$\vec{e}_2 = (0, 1, 0, \dots, 0.)$

$\vec{e}_1 = (1, 0, \dots, 0)$, ...

$, \dots, \quad \vec{e}_n = (0, \dots, 0, 1).$

Then for any vector $(c_1, \dots, c_n) \in K^n$,

we can write

$$(c_1, \dots, c_n) = c_1\vec{e}_1 + c_2\vec{e}_2 + \dots + c_n\vec{e}_n.$$

(In K^3, usually write $\vec{i}, \vec{j}, \vec{k}$).

<u>Ex:</u> $\{1, x, x^2, \dots, x^n\}$ is a basis

for the set of degree at most n polynomials

over K.

<u>Ex</u>: The space of <u>all</u> polynomials over K

is infinite dimensional.

<u>Theorem (Uniqueness of Basis Representation)</u>

If $S = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for the finite-dimensional

K-vector space V, then every $\vec{v} \in V$ can be

expressed in the form

$$\vec{v} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n \qquad (*)$$

in exactly one way.

<u>Proof:</u> Since $\text{span}(S) = V$, every vector can be

expressed in the form $(*)$. Suppose a vector $\vec{v} \in V$

be written

can $\vec{v} = c_1\vec{v}_1 + \cdots + c_n v_n$

and also $\vec{v} = k_1\vec{v}_1 + \cdots + k_n\vec{v}_n$.

Then $\vec{v} - \vec{v} = \vec{0} = (c\vec{v}_1 + \cdots + c_n\vec{v}_n) - (k_1\vec{v}_1 + \cdots + k_n\vec{v}_n)$

So
$$\vec{0} = c_1\vec{v}_1 - k_1\vec{v}_1 + \cdots + c_n\vec{v}_n - k_n\vec{v}_n$$
$$= (c_1 - k_1)\vec{v}_1 + \cdots + (c_n - k_n)\vec{v}_n.$$

Since $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent, $(c_i - k_i) = 0$ for all i. Hence $c_i = k_i$. $\square$.

This uniqueness of representation allows us to formalize what we mean by coordinates.

For example, in $\mathbb{R}^3$, the coordinate $\vec{v} = (a, b, c)$ expresses that $\vec{v} = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$.

<u>Definition</u>: Suppose that $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis of a K-vector space V, and suppose $\vec{v} \in V$ is such that
$$\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n.$$
Then the scalars $c_1, \ldots, c_n \in K$, are called the <u>coordinates of $\vec{v}$ relative to</u> S. The vector $(c_1, \ldots, c_n) \in K^n$ is called the <u>coordinate vector of $\vec{v}$ relative to</u> S. We write

$$(\vec{v})_S = (c_1, \ldots, c_n)$$

or
$$[\vec{v}]_S = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Remark: we tacitly assume that the order of a given vector remains fixed.

4.5 Dimension

Theorem: Let V be a finite dimensional vector space. Then every basis for V has the same size.

Proof: see text book.

Theorem: Let V be a finite dimensional V-space and let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be any basis.

a) If a set $S \subseteq V$ has more than n vectors it is linearly dependent.

b) If S has fewer than n-vectors, $\text{Span}(S) \neq V$.

Definition: The dimension of a finite dimensional vector space V is the size of any basis.
(The zero vector space has dimension 0 by convention).

Ex: $\dim(K^n) = n$.
Ex: $\dim(P_n) = n+1$ since $\{1, x, \dots, x^n\}$ is a basis.

Example: $\dim(\mathrm{Span}(S))$.
suppose $S = \{\vec{v}_1, \dots, \vec{v}_n\}$. If S is linearly independent, then S is a basis for $\mathrm{span}(S)$,

and so $\dim(\mathrm{Span}(S)) = n$.

If S is not lin. indep., then $\dim(\mathrm{Span}(S))$ is the size of the largest independent subset of S.

Plus/minus Theorem Let V be a v-space, $S \subseteq V$, $S \neq \emptyset$.
(+) If $S \subseteq V$ is linearly independent and $\vec{v} \in V \setminus \mathrm{Span}(S)$, then $S \cup \{\vec{v}\}$ is still linearly independent.

(−) Suppose $\vec{v} \in S$ is such that $\vec{v} \in \mathrm{Span}(S - \{\vec{v}\})$.
Then $\mathrm{span}(S) = \mathrm{span}(S - \{\vec{v}\})$.

Pf: (⇒) suppose $S \cup \{\vec{v}\}$ is linearly <u>dependent</u>.

Then there is $\vec{v}_1, \dots \vec{v}_r \in S$ such that

$$c\vec{v} + c_1\vec{v}_1 + \dots + c_n\vec{v}_n = 0.$$

where $c \neq 0$ and at least one of the c_i's is nonzero (since S is independent). But then

$$\vec{v} = \frac{-1}{c}(c_1\vec{v}_1 + \dots c_n\vec{v}_n)$$

so $\vec{v} \in \text{Span}(S)$.

(⇐) Exercise: show $\text{span}(S) \subseteq \text{span}(S - \{\vec{v}\})$. □

<u>Theorem:</u> Suppose $\dim(V) = n$. Let $S \subseteq V$ be a set of size n. Then S is a basis if either

i) $\text{span}(S) = V$

ii) S is linearly indep.