Lecture 6

5.3 Complex Vector spaces..

Defn: A complex vector space V is a K-vector space with $K = \mathbb{C}$.

- We will be mostly concerned with complex n-space $\mathbb{C}^n = \{(c_1, \dots, c_n) ; c_i \in \mathbb{C}\}$.

- Let $\vec{v} = (a_1 + b_1 i, a_2 + b_2 i, \dots, a_n + b_n i) \in \mathbb{C}^n$.
  - Define:

$$Re(\vec{v}) = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$$

$$Im(\vec{v}) = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n.$$

  - We have that

$$\vec{v} = Re(\vec{v}) + i\,Im(\vec{v}).$$

We define the complex conjugate of a vector $\vec{v}$ to be

$$\overline{\vec{v}} = Re(\vec{v}) - i\,Im(\vec{v}).$$

- The same definition applies in the case of matrices with complex entries, $A = Re(A) + i\, Im(A)$.

e.g.: if $A = \begin{bmatrix} 1+i & 3-4i \\ 2i & 42 \end{bmatrix}$

then $\overline{A} = \begin{bmatrix} 1-i & 3+4i \\ -2i & 42 \end{bmatrix}$.

Properties of Complex Conjugation

Let $\vec{u}, \vec{v} \in \mathbb{C}^n$, $k \in \mathbb{C}$, then

1) $\overline{\overline{\vec{v}}} = \vec{v}$
2) $\overline{k\vec{v}} = \overline{k}\,\overline{\vec{v}}$
3) $\overline{\vec{u}+\vec{v}} = \overline{\vec{u}} + \overline{\vec{v}}$.

Let A be an $m \times k$ complex matrix and B be a $k \times n$ complex matrix. Then:

1) $\overline{\overline{A}} = A$
2) $\overline{(A^T)} = (\overline{A})^T$
3) $\overline{AB} = \overline{A}\,\overline{B}$.

<u>Defn</u> Given $\vec{u} = (u_1, \dots, u_n)$, $\vec{v} = (v_1, \dots, v_n) \in \mathbb{C}^n$, define the complex dot (Euclidean) product to be

$$\vec{u} \cdot \vec{v} = u_1 \overline{v_1} + \dots + u_n \overline{v_n}$$

Using this define the Euclidean norm

$$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{|v_1|^2 + \dots + |v_n|^2}$$

We say $\vec{v}$ is a unit vector if $\|\vec{v}\| = 1$.

Theorem: $\vec{u}, \vec{v}, \vec{w} \in \mathbb{C}^n$, $k \in \mathbb{C}$, Then

1) $\vec{u}\cdot\vec{v} = \overline{\vec{v}\cdot\vec{u}}$ (antisymmetry).
2) $\vec{u}(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$. (dist.
3) $k(\vec{u}\cdot\vec{v}) = (k\vec{u})\cdot\vec{v}$.
4) $k(\vec{u}\cdot\vec{v}) = \vec{u}\cdot(\bar{k}\vec{v})$
5) $\vec{v}\cdot\vec{v} \geq 0$, $\vec{v}\cdot\vec{v} = 0$ iff $\vec{v} = \vec{0}$.

Eigenvalues and Eigenvectors

Recall: If A is a complex matrix, and $\vec{v}$ is a vector such that

$$A\vec{v} = \lambda\vec{v}$$

for some $\lambda \in \mathbb{C}$, then we say that $\vec{v}$ is an eigenvector for A, and λ is an eigenvalue corresponding to $\vec{v}$.

Theorem: If A is $n\times n$ complex matrix, and $\lambda \in \mathbb{C}$ a scalar, then TFAE:

1) λ is an eigenvalue of A.
2) λ is a solution of $\det(\lambda I - A) = 0$. (characteristic equation)
3) The system $(\lambda I - A)\vec{x} = \vec{0}$ has a non trivial solution
4) There is a non-zero vector $\bar{x}$ such that
$$A\bar{x} = \lambda \bar{x}.$$

If A is an $n\times n$ matrix, then the set of vectors in $\mathbb{C}^n$ which satisfy $A\vec{x} = \lambda\vec{x}$ is a subspace called the eigenspace of λ.

Ex: Consider the matrix $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$.

The characteristic equation is

$$\det\begin{pmatrix} \lambda-3 & 2 \\ -4 & \lambda+1 \end{pmatrix} = (\lambda-3)(\lambda+1) - (-4)(2)$$
$$= \lambda^2 - 3\lambda + \lambda - 3 + 8$$
$$= \lambda^2 - 2\lambda + 5$$

The solutions to $\lambda^2 - 2\lambda + 5 = 0$ are

$$\lambda = \frac{2 \pm \sqrt{4 - 4(5)}}{2} = \frac{2 \pm 2\sqrt{-4}}{2}$$
$$= 1 \pm 2i$$

Let's find the eigenspace corresponding to $\lambda = 1+2i$:

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (1+2i) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So $$\begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\downarrow$$

$$\begin{bmatrix} 1 & \frac{-2}{2-2i} \\ 4 & -2-2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\downarrow$$

$$\begin{bmatrix} 1 & \frac{-1}{1-i} \\ 0 & \underbrace{-2-2i+\frac{4}{1-i}}_{=0} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -\frac{1}{2}-\frac{1}{2}i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So a non trivial solution is given by

$$x_1 - \frac{1}{2}(1+i)x_2 = 0$$

$$x_1 = \frac{1}{2}(1+i)x_2.$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

So the eigenspace of $\lambda = 1+2i$ is the subspace of $\mathbb{C}^2$ spanned by $\vec{v} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$ (i.e. its a complex line.