Lecture 7

<u>Ex:</u> Consider the matrix $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$.

The characteristic equation is

$$\det\begin{pmatrix} \lambda-3 & 2 \\ -4 & \lambda+1 \end{pmatrix} = (\lambda-3)(\lambda+1)-(-4)(2)$$
$$= \lambda^2 - 3\lambda + \lambda - 3 + 8$$
$$= \lambda^2 - 2\lambda + 5$$

The solutions to $\lambda^2 - 2\lambda + 5 = 0$ are

$$\lambda = \frac{2 \pm \sqrt{4-4(5)}}{2} = \frac{2 \pm 2\sqrt{-4}}{2}$$
$$= 1 \pm 2i$$

Let's find the eigenspace corresponding to $\lambda = 1+2i$:

$\vec{x} = \binom{x_1}{x_2}$

$$\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (1+2i)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

so

$$\begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & \frac{-2}{2-2i} \\ \end{bmatrix}\begin{bmatrix} x_1 \\ \end{bmatrix} = \begin{bmatrix} 0 \\ \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2-2i \end{bmatrix}\begin{bmatrix} x_2 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & \frac{-1}{1-i} \\ 0 & \underbrace{-2-2i+\frac{4}{1-i}}_{=0} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -\frac{1}{2}-\frac{1}{2}i \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving: $x_1 - \frac{1}{2}(1+i)x_2 = 0$

$$x_1 = \frac{1}{2}(1+i)x_2.$$

So the general solution is:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{t}{2}(1+i) \\ t \end{bmatrix}.$$

Taking $t=2$ gives a particular, non-trivial solution:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

So the eigenspace of $\lambda = 1+2i$ is the subspace of $\mathbb{C}^2$ spanned by $\vec{v} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$ (i.e. it's a line.

complex ...

(i.e. $\left\{\begin{bmatrix} 1+i \\ 2 \end{bmatrix}\right\}$ is a basis for the eigenspace

Theorem: Let A be an $n\times n$ real or complex matrix. Suppose that $\vec{v}_1, \dots, \vec{v}_k$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_k$. Then $\vec{v}_1, \dots, \vec{v}_k$ are linearly independent.

Proof: - For a contradiction, assume $\vec{v}_1, \dots, \vec{v}_k$ are linearly dependent.

- we may assume that k is minimal, i.e. for any subset $S \subsetneq \{\vec{v}_1, \dots, \vec{v}_k\}$, S is linearly independent.

  (convince yourself that we can do this!).

- by minimality of k, there are $c_1, \dots, c_k$, all nonzero such that

$$c_1\vec{v}_1 + \dots + c_k\vec{v}_k = \vec{0}.$$

- Now,

$$\vec{0} = A(\vec{0}) = A(c_1\vec{v}_1 + \dots + c_k\vec{v}_k)$$
$$= c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \dots + c_k\lambda_k\vec{v}_k$$

- But also

$$\vec{0} = \lambda_1\vec{0} = \lambda_1(c_1\vec{v}_1 + \dots + c_k\vec{v}_k)$$
$$= c_1\lambda_1\vec{v}_1 + c_2\lambda_1\vec{v}_2 + \dots + c_k\lambda_1\vec{v}_1.$$

-Now

$$\vec{0} = \vec{0} - \vec{0} = (c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \cdots + c_k\lambda_k\vec{v}_k) - (c_1\lambda_1\vec{v}_1 + c_2\lambda_1\vec{v}_1 + \cdots + c_k\lambda_1\vec{v}_1)$$
$$= c_1(\lambda_1 - \lambda_1)\vec{v}_1 + c_2(\lambda_2 - \lambda_1)\vec{v}_2 + \cdots + c_k(\lambda_k - \lambda_1)\vec{v}_k.$$
$$= c_2\underbrace{(\lambda_2 - \lambda_1)}_{\neq 0}\vec{v}_2 + \cdots + c_k\underbrace{(\lambda_k - \lambda_1)}_{\neq 0.}\vec{v}_k$$

- This contradicts the minimality of k. $\square$.

5.2 Diagonalization

Recall: A matrix is diagonal if all of its non-zero entries lay on the main diagonal.

Ex $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is diagonal

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is not diagonal.

- Square matrices A and B are called similar if there is an invertible matrix P such that $B = P^{-1}AP$. (so $A = PBP^{-1}$)
  - ↳ see text for a list of properties preserved by similarity. (its an equivalence relation).

Defn: A square matrix A is called diagonalizable if it is similar to a diagonal matrix.

<u>Theorem:</u> Let A be an $n \times n$ matrix. Then A is diagonalizable iff A has n linearly independent eigenvectors.

<u>Proof:</u> ($\Rightarrow$) Let $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \lambda_n \end{bmatrix}$ and suppose that

$$AP = PD,$$

where $P = [\vec{p}_1 \ \vec{p}_2 \cdots \vec{p}_n]$ is invertible.

Then

$$AP = [A\vec{p}_1 \ A\vec{p}_2 \cdots A\vec{p}_n] = [\lambda_1\vec{p}_1 \ \lambda_2\vec{p}_2 \cdots \lambda_n\vec{p}_n].$$

- Hence $\vec{p}_1 \dots \vec{p}_n$ are eigenvectors of A.
- Since P is invertible, $\vec{p}_1 \dots \vec{p}_n$ must be linearly independent.

($\Leftarrow$) - Suppose A has linearly independent eigenvectors $\vec{p}_1 \dots \vec{p}_n$.

- Let $P = [\vec{p}_1 \ \vec{p}_2 \cdots \vec{p}_n]$.
- Since $\vec{p}_1 \dots \vec{p}_n$ are independent, P is invertible.
- Let λ_i be the eigenvalue corresponding to $\vec{p}_i$.
- Then

$$\begin{aligned} AP &= [A\vec{p}_1 \ A\vec{p}_2 \cdots A\vec{p}_n] \\ &= [\lambda_1\vec{p}_1 \ \lambda_2\vec{p}_2 \cdots \lambda_n\vec{p}_n]. \\ &= \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \lambda_n \end{bmatrix} [\vec{p}_1 \ \vec{p}_2 \cdots \vec{p}_n]. = PD \end{aligned}$$

$\underbrace{}_{D}$

check this is true!

So $AP = PD$ i.e. $P^{-1}AP = D$. $\square$.

Combining this theorem with the one from earlier, we have the following:

<u>Theorem:</u> A matrix with n distinct eigenvalues is diagonalizable.