Lecture 8

Theorem: Let A be an $n \times n$ matrix. Then A is diagonalizable iff A has n linearly independent eigenvectors.

Last time, we showed $\Rightarrow$. Now, let's prove the other direction:

($\Leftarrow$) - Suppose A has linearly independent eigenvectors $\vec{p}_1 \dots \vec{p}_n$.

- Let $P = [\vec{p}_1 \ \vec{p}_2 \cdots \vec{p}_n]$.
- Since $\vec{p}_1 \dots \vec{p}_n$ are independent, P is invertible.
- Let λ_i be the eigenvalue corresponding to $\vec{p}_i$.
- Then

$$AP = [A\vec{p}_1 \ A\vec{p}_2 \cdots A\vec{p}_n]$$
$$= [\lambda_1 \vec{p}_1 \ \lambda_2 \vec{p}_2 \cdots \lambda_n \vec{p}_n].$$
$$= \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \lambda_n \end{bmatrix}}_{D} [\vec{p}_1 \ \vec{p}_2 \cdots \vec{p}_n]. \ \underbrace{= PD}$$

check this is true!

So $AP = PD \Leftrightarrow P^{-1}AP = D$. $\square$

$\smile$.

Combining this theorem with the one from earlier, we have the following:

<u>Theorem:</u> A matrix with n distinct eigenvalues is diagonalizable.

Procedure For Diagonalization

Let A be $n \times n$.

<u>Step 1:</u> Determine if A is diagonalizable.
- ↳ A is diagonalizable iff A has n linearly independent eigenvectors.
- ↳ do this by finding bases for the eigenspace and counting the size of each basis (should sum up to n).

<u>Step 2:</u> If A is diagonalizable let $\vec{p}_1, \dots, \vec{p}_n$ be the linearly independent eigenvectors from step 1.
- ↳ form the matrix $P = [\vec{p}_1 \ \vec{p}_2 \ \cdots \ \vec{p}_n]$.

<u>Step 3</u>: $P^{-1}AP$ will be $\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$ when $A\vec{p}_i = \lambda_i \vec{p}_i$.

The following Theorem is often very useful for computations:

Theorem: If A is an $n\times n$ real matrix and $\lambda \in \mathbb{C}$ an eigenvalue with eigenvector $\vec{v}$, then $\overline{\lambda}$ is an eigenvalue with eigenvector $\overline{\vec{v}}$.

Proof: Let $\lambda, \vec{v}$ be as above.

$$\overline{A\vec{v}} = \overline{(\lambda\vec{v})} = \overline{\lambda}\,\overline{\vec{v}}.$$

But also

$$\overline{A\vec{v}} = \overline{A}\,\overline{\vec{v}}.$$

Since A is real, $\overline{A} = A$, so

$$A\overline{\vec{v}} = \overline{\lambda}\,\overline{\vec{v}} \quad \text{as required. } \square.$$

Ex: - Consider the matrix from the last lecture: $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$.

- we found that the eigenvalues of A were $\lambda = 1 \pm 2i$.

- since A is 2×2, and since A has two distinct eigenvalues A is diagonalizable.

- From last lecture, $\vec{p}_1 = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 1+2i$.

- Since A has real entries $\lambda_2 = 1-2i$ and the eigenvector corresponding to λ_2 is $\vec{p}_2 = \overline{\vec{p}_1} = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$
$= \overline{\lambda_1}$

- Then $P = \begin{bmatrix} 1+i & 1-i \\ 2 & 2 \end{bmatrix}$.

- Since P is 2×2,

$$P^{-1} = \frac{1}{(1+i)2 - 2(1-i)} \begin{bmatrix} 2 & i-1 \\ -2 & 1+i \end{bmatrix}$$

$$= \frac{1}{4i} \begin{bmatrix} 2 & i-1 \\ -2 & 1+i \end{bmatrix}$$

- So $P^{-1}AP = \frac{1}{4i} \begin{bmatrix} 2 & i-1 \\ -2 & 1+i \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1+i & 1-i \\ 2 & 2 \end{bmatrix}$

$$= \frac{1}{4i} \begin{bmatrix} 2 & i-1 \\ -2 & 1+i \end{bmatrix} \begin{bmatrix} -1+3i & -1-3i \\ 2+4i & 2-4i \end{bmatrix}$$

$$= \frac{1}{4i} \begin{bmatrix} -8+4i & 0 \\ 0 & 8+4i \end{bmatrix} = \begin{bmatrix} 2i+1 & 0 \\ 0 & -2i+1 \end{bmatrix}$$

$\begin{bmatrix} \lambda_1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} \lambda_1 & \\ 0 & \lambda_2 \end{bmatrix}$$

as required.

<u>useful:</u> Powers of Diagonalizable matrices.

- Suppose A is diagonalizable,
  $D = P^{-1}AP$.
- Let $k \geq 0$. Then

$$A^k = (PDP^{-1})^k$$

$$= \underbrace{(PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})\cdots(\cancel{P}DP^{-1})}_{k \text{ times.}}$$

$$= P\,D^k P^{-1}$$

For a diagonal matrix

$$D = \begin{pmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & c_n \end{pmatrix}$$

we have $D^k = \begin{pmatrix} c_1^k & & 0 \\ & \ddots & \\ 0 & & c_n^k \end{pmatrix}$.