Lecture #9.

(6.1) Inner Product Spaces.

- In this section, we aim to generalize the real and complex dot product on $\mathbb{R}^n$, $\mathbb{C}^n$ respectively.

Defn: A real inner product space is a real vector space V together with a function $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$ such that $\forall \vec{u}, \vec{v}, \vec{w} \in V, \forall k \in \mathbb{R}$,

i) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$ (symmetry)

ii) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ (additivity)

iii) $\langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle$ (homogeneity)

iv) $\langle \vec{v}, \vec{v} \rangle \geq 0$, $\langle \vec{v}, \vec{v} \rangle = 0$ iff $\vec{v} = \vec{0}$ (positivity).

The purpose of introducing inner product spaces is to generalize the usual dot product on $\mathbb{R}^n$.

Examples:

1) $\mathbb{R}^n$, with $\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \vec{v}$ (the usual dot product)

2) - Let $a, b \in \mathbb{R}$, $a < b$.
   - Let $C^0([a,b])$ be the vector space of continuous, real valued functions on $[a,b]$.
   - For $f, g \in C^0([a,b])$, define
   $$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx.$$
   Then $\langle f, g \rangle$ is an inner product.

Inner products give us a notion of length and distance (and even angle!).

Definition: Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. For a vector $\vec{v} \in V$, define the norm of $\vec{v}$ to be
$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}.$$

For $\vec{u}, \vec{v} \in V$, define the distance between $\vec{u}$ and $\vec{v}$ is

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|.$$

Ex: If $(V, \langle \cdot, \cdot \rangle)$ is $\mathbb{R}^n$ with the usual dot product, then $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$, the usual Euclidean length.

Theorem (6.1.1) Let $(V, \langle \cdot, \cdot \rangle)$ be a real inner product space. Let $\vec{u}, \vec{v}, \vec{w} \in V$, $k \in \mathbb{R}$. Then

1) $\langle \vec{0}, \vec{v} \rangle = 0$
2) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u} + \vec{w} \rangle$.
3) $\langle \vec{u}, k\vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle$.
4) $\langle \vec{u} - \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle - \langle \vec{v}, \vec{w} \rangle$.
5) $\langle \vec{u}, \vec{v} - \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle - \langle \vec{u}, \vec{w} \rangle$.

We can also define what it means to be a complex inner product space:

Defn: Let V be a complex innerproduct space. Then $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ is a (complex) inner product iff

1) $\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$

2) $\langle \vec{u}+\vec{v}, \vec{w} \rangle = \langle \vec{u}+\vec{w} \rangle + \langle \vec{v}+\vec{w} \rangle$.

3) $\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$ for $k \in \mathbb{C}$.

5) For all $\vec{v}$, $\langle \vec{v}, \vec{v} \rangle \in \mathbb{R}^+$, and $\langle \vec{v}, \vec{v} \rangle = 0$ iff $\vec{v} = \vec{0}$.

<u>Example:</u> Let $A = \begin{bmatrix} 2 & i \\ -i & 2 \end{bmatrix}$.

Then the map

$$\langle \cdot, \cdot \rangle : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$$

defined by

$$\langle \vec{v}, \vec{w} \rangle = [\vec{v}]^T A [\overline{\vec{w}}]$$

is an inner product on $\mathbb{C}^2$ (different from the standard complex dot product).

$\vec{v} = (v_1, v_2)$ $\quad \vec{w} = (w_1, w_2)$

$$\langle \vec{v}, \vec{w} \rangle = [v_1, v_2] \begin{bmatrix} 2 & i \\ -i & 2 \end{bmatrix} \begin{bmatrix} \overline{w_1} \\ \overline{w_2} \end{bmatrix}$$

$$= [v_1 \ v_2] \begin{bmatrix} 2\overline{w_1} + i\overline{w_2} \\ -i\overline{w_1} + 2\overline{w_2} \end{bmatrix} = v_1(2\overline{w_1} + i\overline{w_2}) + v_2(-i\overline{w_1} + 2\overline{w_2}).$$

<u>Note:</u> Not every matrix works. We chose a special type of matrix.

- For a complex inner product space $(V, \langle , \rangle)$, we define length in the same way as in the real case. For $\vec{v} \in V$, the norm of $\vec{v}$ is $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$.
- We define the distance between $\vec{u}$ and $\vec{v}$ as

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|.$$

Now, we will assume that an inner prod. space $(V, \langle \cdot, \cdot \rangle)$ is either real or complex.

Theorem (The Cauchy-Schwarz Inequality)
If $(V, \langle , \rangle)$ is an inner product space, $\vec{u}, \vec{v} \in V$, then

$$\underbrace{|\langle \vec{u}, \vec{v} \rangle|}_{\text{real or complex absolute value}} \le \underbrace{\|\vec{u}\| \|\vec{v}\|}_{\text{norm.}}$$

Proof: If $\vec{v} = \vec{0}$, then the inequality is true, so we may assume $\vec{v} \neq \vec{0}$. Let $\lambda = \dfrac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2}$. Then

$$
\begin{aligned}
0 &\le \|\vec{u} - \lambda \vec{v}\|^2 \\
&= \langle \vec{u}, \vec{u} \rangle - \langle \lambda \vec{v}, \vec{u} \rangle - \langle \vec{u}, \lambda \vec{v} \rangle + \langle \lambda \vec{v}, \lambda \vec{v} \rangle \\
&= \langle \vec{u}, \vec{u} \rangle - \lambda \langle \vec{v}, \vec{u} \rangle - \bar{\lambda} \langle \vec{u}, \vec{v} \rangle + \lambda \bar{\lambda} \langle \vec{v}, \vec{v} \rangle \\
&= \langle \vec{u}, \vec{u} \rangle - \lambda \overline{\langle \vec{u}, \vec{v} \rangle} - \bar{\lambda} \langle \vec{u}, \vec{v} \rangle + \lambda \bar{\lambda} \|\vec{v}\|^2 \\
&= \|\vec{u}\|^2 - \lambda \overline{\langle \vec{u}, \vec{v} \rangle} - \bar{\lambda} \langle \vec{u}, \vec{v} \rangle + \lambda \bar{\lambda} \|\vec{v}\|^2 \\
&= \|\vec{u}\|^2 - \frac{|\langle \vec{u}, \vec{v} \rangle|^2}{\|v\|^2} - \frac{|\langle \vec{u}, \vec{v} \rangle|^2}{\|v\|^2} + \frac{|\langle \vec{u}, \vec{v} \rangle|^2}{\|\vec{v}\|^2} \\
&= \|\vec{u}\|^2 - \frac{|\langle \vec{u}, \vec{v} \rangle|^2}{\|v\|^2}
\end{aligned}
$$

So $$|\langle \vec{u}, \vec{v} \rangle|^2 \leq \|\vec{u}\|^2 \|\vec{v}\|^2$$

and hence $|\langle \vec{u}, \vec{v} \rangle|^2 \leq \|\vec{u}\| \|\vec{v}\|$ $\square$.