Mathematics 2R3 Test 1

Dr. Greg Cousins Feb. 4, 2020

Last Name:______________________________ Initials:__________

Student No.:______________________________

- The test is 50 minutes long.
- The test has 6 pages and 5 questions and is printed on BOTH sides of the paper.
- You are responsible for ensuring that your copy of the paper is complete. Bring any discrepancies to the attention of the invigilator.
- Attempt all questions and write your answers in the space provided.
- Marks are indicated next to each question; the total number of marks is 25.
- You may use a McMaster standard Casio fx-991 MS or MS Plus calculator (no communication capability); no other aids are not permitted.
- Use pen to write your test. If you use a pencil, your test will not be accepted for regrading (if needed).

Good Luck!

Score

Question	1	2	3	4	5	Total
Points	5	5	5	5	5	25
Score						

continued ...

1. (5 marks) Put your answer in the space provided for each part.

   (a) Every real $n \times n$ matrix has a real eigenvalue. True or false.
   False: See question 3 for a real matrix with no real eigenvalues.

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   (b) If $\dim(V) = n$, then every set of $n+1$ vectors is linearly dependent. True or false.
   True: If not, then there is a basis of size $n+1$, which contradicts the fact that $\dim(V) = n$.

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   (c) The set of polynomials with real coefficients is a finite dimensional vector space. True or false.
   False: $1, x, x^2, \ldots$ are all linearly independent. This is in contrast to the vector space P_n of polynomials of degree at most n, which has a basis $\{1, x, \ldots, x^n\}$, making $\dim(P_n) = n+1$.

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   (d) If $u = (i, 1)$ and $v = (1, i)$ in $\mathbf{C}^2$ then $u \cdot v =$ __________
   $u \cdot v = (i)(1) + (-i)(1) = i - i = 0.$

   (e) Let A be a *real*, 4×4 matrix with eigenvalues λ_1, λ_2, λ_3, and λ_4. If $\lambda_1 = 42 - \sqrt{17}i$ and $\lambda_2 = 137 + 57i$, what are λ_2 and λ_3?
   Remark: There was a typo in this question. It should have read "what are λ_3 and λ_4". In that case the two other eigenvalues are the complex conjugates of the first two, since A is a real matrix.

continued ...

2. (a) (2 marks) Let $z = \sqrt{3} + \sqrt{3}i$. Write z^2 in polar form.

   Solution: Let $z = \sqrt{3}(1+i)$. Then $z^2 = 3(1+i)^2$. We have that

$$1 + i = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$$

   and so

$$z^2 = 6\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$$

   (b) (3 marks) Suppose V is a complex inner product space and $\mathbf{u}, \mathbf{v} \in V$ such that $\langle \mathbf{u}, \mathbf{u} \rangle = 1$, $\langle \mathbf{u}, \mathbf{v} \rangle = i - 1$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 1$. Compute $\|\mathbf{u} + \mathbf{v}\|$. What can you conclude about $\mathbf{u} + \mathbf{v}$?

Solution: We compute $\|\mathbf{u} + \mathbf{v}\|^2$.

$$\begin{aligned}
\|\mathbf{u} + \mathbf{v}\|^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle \\
&= \langle \mathbf{u}, \mathbf{u} + \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle \\
&= \overline{\langle \mathbf{u} + \mathbf{v}, \mathbf{u} \rangle} + \overline{\langle \mathbf{u} + \mathbf{v}, \mathbf{v} \rangle} \\
&= \overline{\langle \mathbf{u}, \mathbf{u} \rangle} + \overline{\langle \mathbf{v}, \mathbf{u} \rangle} + \overline{\langle \mathbf{u}, \mathbf{v} \rangle} + \overline{\langle \mathbf{v}, \mathbf{v} \rangle} \\
&= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \overline{\langle \mathbf{u}, \mathbf{v} \rangle} + \langle \mathbf{v}, \mathbf{v} \rangle \\
&= 1 + (i - 1) + \overline{(i - 1)} + 1 \\
&= 1 + (i - 1) + (-i - 1) + 1 \\
&= 1 + 1 - 1 - 1 + i - i \\
&= 0.
\end{aligned}$$

Hence $\|\mathbf{u} + \mathbf{v}\| = 0$. By the positivity axiom, $\mathbf{u} = -\mathbf{v}$.

3. (5 marks) Suppose A is the matrix

$$\begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}.$$

Find a matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

Solution: First, we should find the eigenvalues. We compute the characteristic polynomial:

$$\begin{aligned} \det(\lambda I - A) &= \left| \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} \right| \\ &= \begin{vmatrix} \lambda - 3 & -5 \\ 2 & \lambda + 3 \end{vmatrix} \\ &= (\lambda - 3)(\lambda + 3) - (2)(-5) \\ &= \lambda^2 - 3\lambda + 3\lambda - 9 + 10 \\ &= \lambda^2 + 1. \end{aligned}$$

Therefore $\det(\lambda I - A) = 0$ iff $\lambda = \pm i$.

Next, let's compute the eigenspace for $\lambda = i$. We want to solve the equation

$$\begin{pmatrix} i - 3 & -5 \\ 2 & i + 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or, equivalently,

$$\begin{aligned} (i - 3)x - 5y &= 0 \\ 2x + (i + 3)y &= 0. \end{aligned}$$

Solving for x in the second equation, we have

$$x = \frac{-(i+3)}{2} y.$$

Note that if we substitute this x into the first equation, we get

$$\begin{aligned} 0 &= (i - 3)\frac{-(i+3)}{2} y - 5y \\ &= \frac{-(-1 + 3i - 3i - 9)}{2} y - 5y \\ &= \frac{10}{2} y - 5y \\ &= 5y - 5y = 0, \end{aligned}$$

so $0 = 0$. This equation tells us no information. It follows that the equations are multiples of one another. Therefore the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{-(i+3)}{2} \\ 1 \end{pmatrix} t$$

, for $t \in \mathbb{C}$. This means that

$$\left\{ \begin{pmatrix} \frac{-(i+3)}{2} \\ 1 \end{pmatrix} \right\}$$

is a basis for the $\lambda = i$ eigenspace. To find a basis for the $\lambda = -i$ eigenspace, observe that since A is real, we only need to take the complex conjugate. Hence

$$\left\{ \begin{pmatrix} \frac{-3+i}{2} \\ 1 \end{pmatrix} \right\}$$

is a basis for the $\lambda = -i$ eigenspace. From here, we are essentially done. Let

$$P = \begin{pmatrix} \frac{-(i+3)}{2} & \frac{-3+i}{2} \\ 1 & 1 \end{pmatrix}.$$

Then

$$D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

4. Let $V = C^1([0, \pi])$ be the real vector space of continuous, real-valued functions on $[0, \pi]$.

(a) (3 marks) Verify that the operation defined by

$$\langle f, g \rangle := \int_0^\pi f(x)g(x)\sin(x)dx$$

is an inner product on V.

Solution: we need only to verify that $\langle \cdot, \cdot \rangle$ satisfies the axioms of a real inner-product.

i. Symmetry:

$$\begin{aligned} \langle f, g \rangle &= \int_0^\pi f(x)g(x)\sin(x)dx \\ &= \int_0^\pi g(x)f(x)\sin(x)dx \\ &= \langle g, f \rangle. \end{aligned}$$

ii. Linearity in the first variable:

$$\begin{aligned} \langle f + g, h \rangle &= \int_0^\pi (f(x) + g(x))h(x)\sin(x)dx \\ &= \int_0^\pi f(x)h(x)\sin(x) + g(x)h(x)\sin(x)dx \\ &= \int_0^\pi f(x)h(x)\sin(x)dx + \int_0^\pi g(x)h(x)\sin(x)dx \\ &= \langle f, h \rangle + \langle g, h \rangle. \end{aligned}$$

iii. Homogeneity: for k a scalar, we have

$$\begin{aligned} \langle kf, g \rangle &= \int_0^\pi kf(x)g(x)\sin(x)dx \\ &= k\int_0^\pi f(x)g(x)\sin(x)dx \\ &= k\langle f, g \rangle. \end{aligned}$$

iv. Positivity: Let $f \in C[0, \pi]$. Then

$$\langle f, f \rangle = \int_0^\pi f(x)^2 \sin(x)dx.$$

Since $f(x)^2$ is non-negative, and since $\sin(x)$ is non-negative on the interval $[0, \pi]$, and hence $\langle f, f \rangle \geq 0$. If $f = 0$ then $\langle f, f \rangle = 0$.

Suppose $f(x)$ is non-zero. Then, by continuity, there is $a \in [0, \pi]$ and $\epsilon > 0$ such that $\sin(x)f(x)^2 \gneq 0$ on $(a - \epsilon, a + \epsilon)$. Then

$$0 \lneq \int_{a-\epsilon}^{a+\epsilon} f(x)^2 \sin(x)dx \leq \langle f, f \rangle.$$

(b) (2 marks) With respect to this inner product, show that

$$\|\sin(x)+\cos(x)\|^2 = \|\sin(x)\|^2 + \|\cos(x)\|^2.$$

Solution: The idea here was to recognize the equation as an instance of the generalized Pythagorean theorem. The result then follows from showing that $\sin(x)$ and $\cos(x)$ are orthogonal with respect to the given inner product.

$$\langle \sin(x), \cos(x) \rangle = \int_0^\pi \sin^2(x)\cos(x)dx.$$

Making the substitution $u = \sin(x)$, $du = \cos(x)dx$, we have

$$\begin{aligned}\langle \sin(x), \cos(x) \rangle &= \int_0^\pi \sin^2(x)\cos(x)dx \\ &= \int_{\sin(0)}^{\sin(\pi)} u^2 du \\ &= \int_0^0 u^2 du = 0.\end{aligned}$$

Therefore $\sin(x)$ and $\cos(x)$ are orthogonal with respect to this inner product, and so

$$\|\sin(x)+\cos(x)\|^2 = \|\sin(x)\|^2 + \|\cos(x)\|^2.$$

5. Suppose that W is a subspace of an inner product space V.

   (a) (2 marks) Show that $W \subseteq (W^\perp)^\perp$.

   Solution: We want to show that W is a subset of $(W^\perp)^\perp$. Suppose that $\mathbf{w} \in W$. Then $\mathbf{w}$ is also an element of $(W^\perp)^\perp$ if $\mathbf{w}$ is orthogonal to every element $W^\perp$ (by definition of orthogonal complement). Let $\mathbf{v} \in W^\perp$. Then $\langle \mathbf{w}, \mathbf{v} \rangle = 0$, since $\mathbf{w} \in W$. This shows that $W \subseteq (W^\perp)^\perp$.

   (b) (3 mark) Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ be a basis for W. Prove that

$$W^\perp = \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{w}_i \rangle = 0, i = 1, \ldots, n\}.$$

   Solution: Lets say

$$X = \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{w}_i \rangle = 0, i = 1, \ldots, n\}.$$

   We want to show that $W^\perp = X$. It is easy to see that $W^\perp \subseteq X$; if $\mathbf{v} \in W^\perp$, then $\mathbf{v}$ is orthogonal to *every* vector in W and so, in particular, $\mathbf{v}$ is orthogonal to each of $\mathbf{w}_1, \ldots, \mathbf{w}_n$.

   On the other hand, to see that $X \subset W^\perp$, we let $\mathbf{v} \in X$ be arbitrary. We need to show that $\mathbf{v}$ is orthogonal to *every* vector in W (and so is in $W^\perp$ by definition). Let $\mathbf{w} \in W$ be arbitrary. Since $\mathbf{w}_1, \ldots, \mathbf{w}_n$ is a basis for W, we may write

$$\mathbf{w} = c_1\mathbf{w}_1 + \ldots + c_n\mathbf{w}_n$$

   for some scalars $c_1, \ldots, c_n$. Then

$$\begin{aligned} \langle \mathbf{w}, \mathbf{v} \rangle &= \langle c_1\mathbf{w}_1 + \ldots + c_n\mathbf{w}_n, \mathbf{v} \rangle \\ &= \langle c_1\mathbf{w}_1, \mathbf{v} \rangle + \ldots + \langle c_n\mathbf{w}_n, \mathbf{v} \rangle \\ &= c_1\langle \mathbf{w}_1, \mathbf{v} \rangle + \ldots + c_n\langle \mathbf{w}_n, \mathbf{v} \rangle. \end{aligned}$$

   Now, since $\mathbf{v} \in X$, we have

$$\langle \mathbf{v}, \mathbf{w}_i \rangle = 0 = \langle \mathbf{w}_i, \mathbf{v} \rangle$$

   for all $1 \leq i \leq n$, and so

$$\langle \mathbf{w}, \mathbf{v} \rangle = 0.$$

   Since $\mathbf{w} \in W$ was arbitrary, we have shown that $\mathbf{v} \in W^\perp$, and so $X \subseteq W^\perp$.

   Since $X \subseteq W^\perp$ and $W^\perp \subseteq X$, we have that $W^\perp = X$, as required.

THE END