Lecture 5

Ex: The Matching Problem Example

This ~~is~~ example 5m from the textbook.

Idea: Suppose that N people throw their hats into into the centre of the room. The hats are thoroughly mixed and each person picks up a hat. What is the probability that no one gets their own hat? $\underbrace{\text{no one gets their own hat}}_{\text{Event } A}$

Let's consider some cases:

- $N=1$: One person, one hat. He gets his own hat. $P(A)=0$.
- $N=2$: two people two hats. The hats can either switch, or go to their respective owners, so $P(A) = 1/2$.

So the probability seems to be increasing.

For general N: we compute $P(A^c)$

A^c = "at least one person gets their own hat.

Let E_i be the event:

E_i = the i^{th} person gets their own hat.

Therefore, $A^c = \bigcup_{i=1}^{N} E_i$ [and $P(A) = 1 - P(A^c)$.]

By the inclusion-exclusion principle:

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{r=1}^{N} (-1)^{r+1} \sum_{i_1 < \dots < i_r} P(E_{i_1} \dots E_{i_r})$$

$$= P(E_1) + \dots + P(E_N) + \dots + (-1)^{N+1} P(E_1 \dots E_N).$$

Note that, for each r, the sum

$$\sum_{i_1 < \dots < i_r} P(E_{i_1} \dots E_{i_r})$$

has $\binom{N}{r}$-many terms.

Let $i_1, \dots, i_r$ be given/chosen.

$P(E_{i_1} \dots E_{i_r})$ is the ~~the~~ $i_1^{th}, \dots, i_r^{th}$ people ~~man~~ all get their own hats. That means there are $n-r$ other people to receive hats, and so these hats can be permuted in $(N-r)!$ many ways. Since there are N hats total, there are $N!$ many ways to permute the hats, hence:

$$P(E_{i_1},\dots,E_{i_r}) = \frac{(N-r)!}{N!}$$

Thus:

$$\sum_{i_1<\dots<i_r} P(E_{i_1}\dots E_{i_r}) = \binom{N}{r}\frac{(N-r)!}{N!}$$

$$= \frac{\cancel{N!}}{r!\,\cancel{(N-r)!}}\cdot\frac{\cancel{(N-r)!}}{\cancel{N!}}$$

$$= \frac{1}{r!}$$

so $$P(A^c) = \sum_{r=1}^{N} (-1)^{r+1}\frac{1}{r!}$$

so $$P(A) = 1 - \sum_{r=1}^{N}(-1)^{r+1}\frac{1}{r!}$$

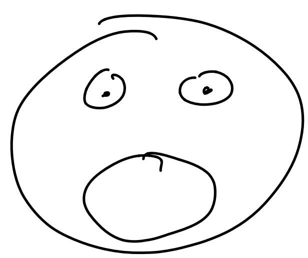
$$= 1 + \sum_{r=1}^{N}(-1)^{r}\frac{1}{r!}$$

r, not r+1. Distribute the (-1)

$$= \sum_{r=0}^{N}\frac{(-1)^r}{r!}$$

So as $N\to\infty$

$$P(A)\to\frac{1}{e}.$$



Probability as a Continuous Set Function

Recall that if S is a finite sample space, then for $E \subseteq S$
$$P(E) = \sum_{x \in E} P(\{x\}).$$
and so, if $F \subseteq E$ and $P(F) = P(E)$, then $F = E$.

This sort of reasoning quickly breaks down, especially in the uncountable case:

Imagine the experiment where you randomly select a real number from $[0,1]$.
What is the probability $P(x \in [0, 1/2])$? Since $[0, 1/2]$ is "half" of the sample space, our intuition tells us that $P(x \in [0, 1/2])$ should be $1/2$. But what about $P(x \in (0, 1/2])$?
The set $(0, 1/2]$ is a proper subset of $[0, 1/2]$.
So the probability "should" be less... but it isn't.

Question: what is $P(x = 0)$?

Defn: We say that a sequence of events $\{E_n : n \geq 0\}$ is increasing if

$$E_0 \subseteq E_1 \subseteq \dots \subseteq E_n \subseteq \dots \quad \left(\begin{array}{c}\text{strictly increasing} \\ \text{if } E_i \subsetneq E_{i+1}, \forall i\end{array}\right),$$

and decreasing if

$$E_0 \supseteq E_1 \supseteq \dots \supseteq E_n \supseteq \dots$$

If $\{E_n\}$ is an increasing sequence, we define

$$\lim_{n\to\infty} E_n = \bigcup_{n=0}^{\infty} E_n$$

I $\{E_n\}$ is decreasing, we say

$$\lim_{n\to\infty} E_n = \bigcap_{n=0}^{\infty} E_n$$

Proposition If $\{E_n\}$ is increasing or decreasing then

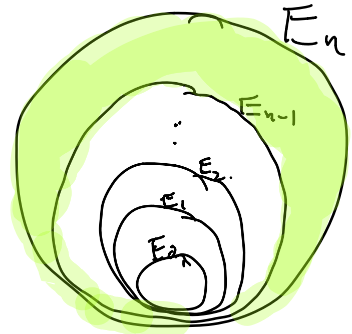
$$P(\lim_n E_n) = \lim_n P(E_n).$$

Pf: We will prove it in the case of increasing.

For each n, let

$$F_n = E_n\left(\bigcup_{i=0}^{n-1} E_i\right)^c$$

$$F_0 = E_0.$$



So
$$\bigcup_{i=0}^{n} F_i = \bigcup_{i=0}^{n} E_n = \underbrace{E_n}_{\text{since increasing}}.$$

and hence $\bigcup_{i=0}^{\infty} F_i = \bigcup_{i=0}^{\infty} E_i$.

<u>But</u> the F_i are mutually exclusive,

so by the axioms:

$$
\begin{aligned}
P\left(\bigcup_i^{\infty} E_i\right) &= P\left(\bigcup_i^{\infty} F_i\right) \\
&= \sum_i^{\infty} P(F_i) \quad \text{(by the last axiom)} \\
&= \lim_{n\to\infty} \sum_{i=0}^{n} P(F_i) \\
&= \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} F_i\right) \\
&= \lim_{n\to\infty} P(E_n). \quad \blacksquare
\end{aligned}
$$

Now, consider the following increasing sequence

$E_i = [1/i,\ 1] \subseteq [0,1]$. $(i = 1, 2, \ldots)$

In the game of choosing a random # in $[0,1]$

$P(x \in E_i)$ is $1 - 1/i$

So $\lim_{n\to\infty} P(x \in E_n) = \lim_{n\to\infty} 1 - \frac{1}{n} = 1$.

<u>But</u> $E = \lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n = (0, 1]$.

So $P(E^c) = 1 - 1 = 0$.

and $E^c = \{x \in [0,1] : x \notin (0,1]\}$

i.e. $E^c = \{0\}$.

Hence $P(x = 0) = 0$.

What about for any real $r \in [0,1]$?

What is $P(x = r)$?