Lecture # 10:

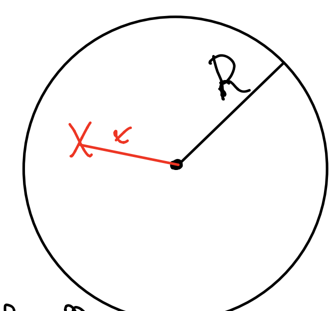
Continuous Random Variables

Recall that a random variable X is said to be <u>continuous</u> (i.e. X is a CRV) if range(X) is an interval of real numbers, (either finite length or infinite).

<u>Examples:</u>

- X = "length of a randomly selected telephone call"
- X = "Time until failure of a machine"
- X = " Electrical current across

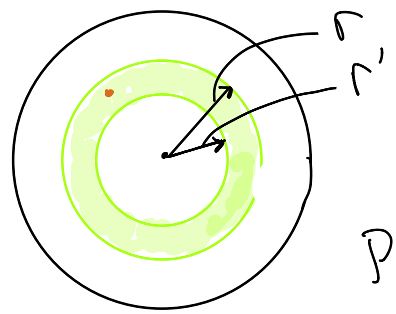
<u>Ex:</u> Consider a dart board of some given radius R



Let X = "distance of dart from centre"
so range(X) $= [0, R]$.

For any particular value $P(X=r)=0$.

To see this,



so the probability of landing between r and r' is

$$P(r' \le x \le r) = \frac{\cancel{\pi} r^2 - \cancel{\pi} r'^2}{\cancel{\pi} R^2}$$

As $r' \to r$

$$P(r' \le x \le r) \to \lim_{r' \to r} \frac{r^2 - r'^2}{R} = 0.$$

In general, for a continuous RV, $P(X=a)=0$ for any particular value of a.

- Recall that if X is <u>discrete</u>, then

for the probability mass function:

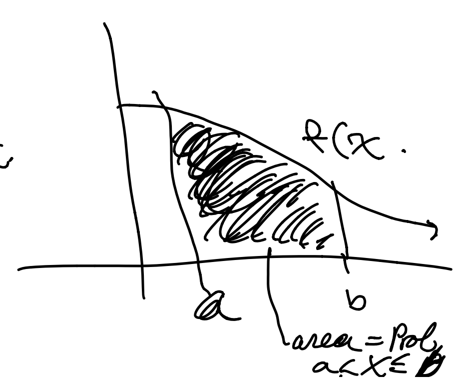
$$P(X=a) = f(a)$$

↳ more generally, for $a \le b \in$ range X,

$$P(a \le X \le b) = \sum_{x=a}^{b} f(x).$$

For a continuous random variable X, we still have a probability "mass" function $f(x)$, and we have

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$



$f(x)$ is called:

Defn Let X be a continuous random variable. Then a **probability density function** $f(x)$ for X is a function such that

1) $f(x) \geq 0 \quad \forall x$
2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$
3) $P(a \leq X \leq b) = \int_a^b f(x)\,dx$.

Note that since $P(X=a)=0$ for any particular a, we have, forall $x_1 \leq x_2$,

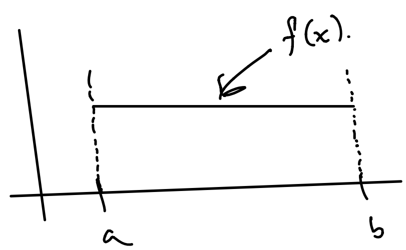
$$P(x_1 \leq X \leq x_2) = P(x_1 < X \leq x_2) = P(x_1 \leq X < x_2) = P(x_1 < X < x_2).$$

Ex: (Uniform Distribution).

Consider an interval $[a,b] \subseteq \mathbb{R}$ (finite length)

Then the function

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } x \in [a,b] \\ 0 & \text{otherwise.} \end{cases}$$



$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} 0\,dx + \int_a^b \frac{1}{b-a}\,dx + \int_b^{\infty} 0\,dx$$

$$= \frac{1}{b-a}\int_a^b dx = \frac{1}{b-a}(b-a) = 1.$$

Ex: Suppose that X is current in a copper wire and the current ranges uniformly between 3.6 and 4.1.

Q: What is $P(3.7 < X < 3.8)$?

A: Since we have a uniform distribution,

$$f(x) = \begin{cases} \frac{1}{4.1-3.6} = 2 & \text{for } 3.6 \le x \le 4.1 \\ 0 & \text{o/w.} \end{cases}$$

So $P(3.7 < X < 3.8) = \int_{3.7}^{3.8} 2\,dx = 2\int_{3.7}^{3.8} dx = 2[3.8-3.7]$
$= 2(0.1)$
$= 0.2.$

Ex: Waiting time in hours at a hospital for admission to the emergency room is given by by a distribution with p.d.f. given by

$$f(x) = \begin{cases} \frac{e^{-\frac{x}{2}}}{2} & \text{for } x \geq 0. \\ 0 & \text{for } x < 0. \end{cases}$$

What is $P(1 < X < 2) = ?$

$$P(1 < x < 2) = \int_1^2 \frac{e^{-\frac{x}{2}}}{2}\,dx = -e^{\frac{-x}{2}}\Big|_1^2 = -e^{-1} + e^{-1/2}$$
$$= 0.2386$$

$$P(X > 3) = 1 - P(X < 3)$$
$$= \lim_{t\to\infty} \int_3^t \frac{e^{-\frac{x}{2}}}{2}\,dx = \lim_{t\to\infty}\left[e^{-3/2} - \cancel{e^{-\frac{t}{2}}}^{\to 0}\right] = e^{-3/2}.$$

Let X be a CRV with p.d.f. $f(x)$.

How do we compute $P(X \le x)$?

By the fundamental theorem of calculus,

$$\frac{d}{dx}\underbrace{\int_{-\infty}^{x} f(t)\,dt}_{F(x) = \text{ antiderivative of } f(x)} = f(x)$$

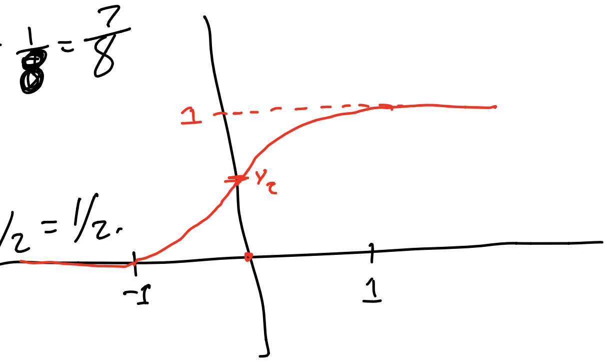
We call $F(x)$ the <u>cumulative distribution function</u> of X. $F(a) = P(X \le a) = \int_{-\infty}^{a} f(x)\,dx$.

We have $P(a < X < b) = F(b) - F(a)$.

<u>Ex</u>: Suppose the cdf of X is given by

$$F(X) = \begin{cases} 0 & \text{for } x \le -1 \\ 1/2\,(x+1)^2 & \text{for } -1 \le x \le 0. \\ 1 - \frac{1}{2}(x-1)^2 & \text{for } 0 < x < 1 \\ 1 & \text{for } x \geq 1. \end{cases}$$

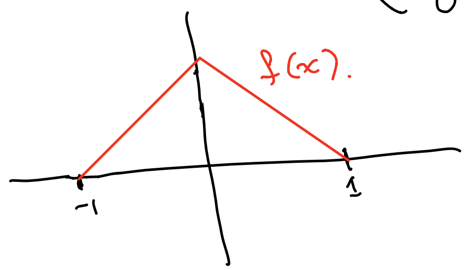
$$P(X \le 1/2) = F(1/2) = 1 - \frac{1}{2}\left(\frac{1}{2} - 1\right)^2 = 1 - \frac{1}{8} = \frac{7}{8}$$



$$P(X > 0) = 1 - P(X \le 0) = 1 - P(0) = 1 - \frac{1}{2}(0+1)^2 = 1 - 1/2 = 1/2.$$

what is the pdf?

$$f(x) = \frac{d}{dx}F(x) = \begin{cases} 0 & \text{for } x \le -1 \\ x+1 & \text{for } -1 < x \le 0. \\ x-1 & \text{for } 0 < x < 1 \\ 0 & \text{for } x \ge 1. \end{cases}$$



Mean and Variance of a Continuous Random Variable

In the continuous setting we can also compute the mean and variance of a CRV X. The transition from discrete to continuous requires us to pass from $\sum$ to $\int$:

If X is a CRV with p.d.f. $f(x)$, then the mean is

$$\mu = E[X] = \int_{-\infty}^{\infty} x f(x)\,dx.$$

For a function $h(X)$ of a random var, we have

$$E(h(X)) = \int_{-\infty}^{\infty} h(x) f(x)\,dx.$$

The variance is.

$$\sigma^2 = V(X) = E[(X-\mu)^2]$$
$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx. \quad \left(= E[X^2] - (E[X])^2 \right.$$