Lecture # 11.

Unfinished from last time:

Let X be a CRV with pdf. $f(x)$.

How do we compute $P(X \leq x)$?

By the fundamental theorem of calculus,

$$\frac{d}{dx} \int_{-\infty}^{x} f(t)\,dt = f(x)$$

$F(x) =$ antiderivative of $f(x)$

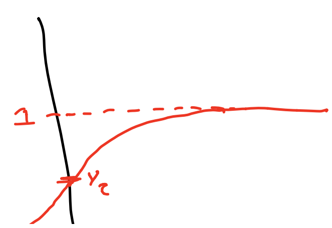
We call $F(x)$ the cumulative distribution function of X. $F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)\,dx$.

We have $P(a < X < b) = F(b) - F(a)$.

Ex: Suppose the cdf of X is given by

$$F(X) = \begin{cases} 0 & \text{for } x \leq -1 \\ \frac{1}{2}(x+1)^2 & \text{for } -1 \leq x \leq 0 \\ 1 - \frac{1}{2}(x-1)^2 & \text{for } 0 < x < 1 \\ 1 & \text{for } x \geq 1. \end{cases}$$

$$P(X \leq 1/2) = F(1/2) = 1 - \frac{1}{2}\left(\frac{1}{2} - 1\right)^2 = 1 - \frac{1}{8} = \frac{7}{8}$$

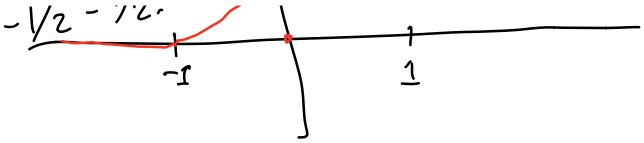


$$P(X > 0) = 1 - P(X \leq 0)$$

$= 1 - F(0) \quad 1 - 1/2$

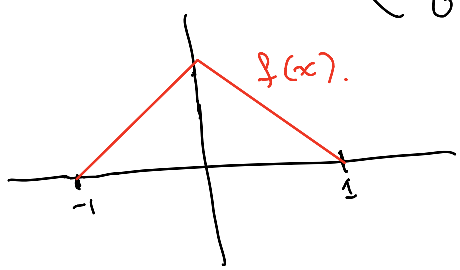
$= 1 - F(0)$

$= 1 - \frac{1}{2}(0+1)^2 = 1 - 1/2 = 1/2.$



what is the pdf ?

$$f(x) = \frac{d}{dx}F(x) = \begin{cases} 0 & \text{for } x \le -1 \\ x+1 & \text{for } -1 < x \le 0 \\ 1-x & \text{for } 0 < x < 1 \\ 0 & \text{for } x \ge 1 \end{cases}$$



Mean and Variance of a Continuous Random Variable

In the continuous setting we can also compute the mean and variance of a CRV X.

The transition from discrete to continuous requires us to pass from Σ to $\int$:

If X is a CRV with p.d.f. $f(x)$, then the mean is

$$\mu = E[X] = \int_{-\infty}^{\infty} x f(x)\, dx.$$

For a function $h(X)$ of a random var, we have

$$E(h(X)) = \int_{-\infty}^{\infty} h(x) f(x)\, dx.$$

The variance is.

$$\sigma^2 = V(X) = E[(X-\mu)^2]$$
$$= \int_{-\infty}^{\infty} (X-\mu)^2 f(x)\, dx.$$

Note that

$$\int_{-\infty}^{\infty} (X-\mu)^2 f(x)\,dx = \int_{-\infty}^{\infty} (x^2 - 2\mu x + \mu^2)\, f(x)\,dx$$

$$= \int_{-\infty}^{\infty} x^2 f(x) - 2\mu x f(x) + \mu^2 f(x)\,dx$$

$$= \underbrace{\int_{-\infty}^{\infty} x^2 f(x)\,dx}_{\substack{= E[X^2] \text{ by} \\ \text{prev. remark.}}} - 2\mu \underbrace{\int_{-\infty}^{\infty} x f(x)\,dx}_{= \mu \text{ by defn.}} + \mu^2 \underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{= 1 \text{ by defn.}}$$

$$= E[X^2] - 2\mu^2 + \mu^2$$

$$Var(X) = E[X^2] - E[X]^2$$

As in the discrete case, $\sigma = \sqrt{\sigma^2} = \sqrt{V(X)}$ is the standard deviation.

Normal Distribution

The normal distribution is one of the most commonly used distributions. The normal distribution is likely very familiar already to many of you.

If an experiment is repeated many times, then the averages of the repetitions tend to have a normal distribution.
We will talk about this more precisely in chapter 5, but what we mean is that averages of "similar" RV's tend to a normal distribution (central limit theorem).

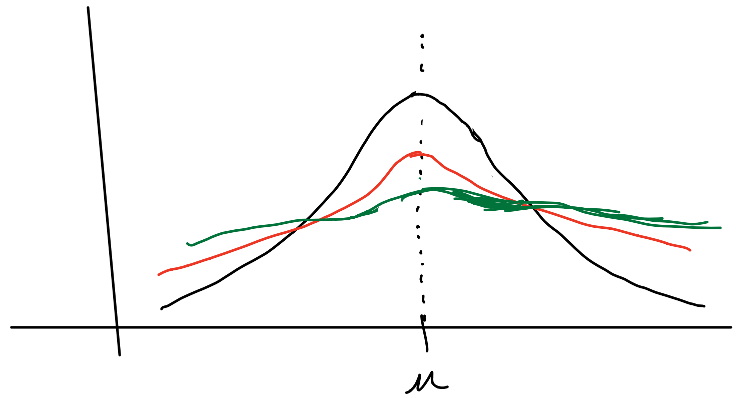
The normal distribution has a p.d.f. given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

where μ and σ are parameters with $\mu, \sigma \in \mathbb{R}$, $\sigma > 0$.

In particular, if X is normal with parameters μ, σ^2 (we write $X = N(\mu, \sigma^2)$) then

$$E[X] = \mu \quad \text{and} \quad V(X) = \sigma^2.$$



↑ σ^2 increase, bell-shaped curve gets "flatter"/more spread out.

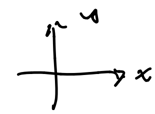
<u>Trouble:</u> $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$

has <u>no</u> <u>antiderivative</u>!

So we need to lookup values of

$P(N(\mu,\sigma^2) \le x)$ in a chart.

<u>Exercise:</u> Find the value of $\int_{-\infty}^{\infty} e^{-x^2}$.



<u>Hint:</u>

Integrate $\int_{-\infty}^{\infty} e^{-x^2}dx$ by integrating $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy$

by a polar change of variables $r = x^2+y^2$

$dxdy = r\,dr\,d\theta$ (Then observe that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2$

Since different values of μ, σ give different normal distributions, it is inconvenient to have tables for everything. Thus, we want to "standardize" to a particular one.

<u>Def</u>: The <u>standard</u> <u>normal</u> distribution is

$$Z = N(0, 1).$$

We denote the cdf by

$$\underbrace{\Phi(z)= P(Z < z)}.$$

these are the "z-scores" you look up in tables.

Fortunately, given a normal random variable $X = N(\mu, \sigma^2)$, the random variable

$$Z = \frac{X-\mu}{\sigma}$$

is the standard normal random var.

Observe that

$$X \leq x \text{ iff } \frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}$$

and so if we want to know $P(X \le x)$, then

$$P(X \le x) = P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right)$$

$$= P\left(Z \le \frac{x-\mu}{\sigma}\right)$$

$$= \underbrace{\Phi\left(\frac{x-\mu}{\sigma}\right)}_{\text{can look this up in a table.}}$$