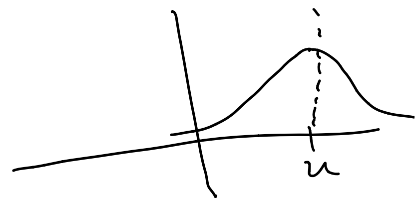
Lecture # 12.

More on the normal Distribution.

Recall from last time:

$X = N(\sigma^2, \mu)$ is the normal distribution with mean μ and variance σ^2



- X has cumulative dist. function

$$P(X \le x) = \int_{-\infty}^{\infty} \underbrace{\frac{e^{\frac{-(t-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}}_{\text{no-antiderivative.}} \, dt$$

- the variable $Z = N(0,1)$ is called the "Standard normal"

- Given any other normal RV $X = N(\mu, \sigma^2)$, we can "standardize":

$$P(X \le x) = P\Big(Z \le \underbrace{\frac{x-\mu}{\sigma}}_{z}\Big) = \Phi\left(\frac{x-\mu}{\sigma}\right) = \Phi(z)$$

Let's do some examples of using Normal distributions.

Example: File transfer from server to computer is normally distributed with mean 5.75 Mbps and variance $(0.35)^2$.

a) What is the prob. that the transfer speed is ≥ 6.7 Mbps?

$$P(X \geq 6.7) = 1 - P(X < 6.7)$$
$$= 1 - P\left(Z < \frac{6.7 - 5.75}{0.35}\right)$$
$$= 1 - P(Z < 2.71)$$
$$= 1 - \underbrace{\Phi(2.71)}_{\text{look this up}}$$
$$= 1 - 0.996636 = 0.003364.$$

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b) What is Prob. that the speed is ≤ 5.5 Mbps?

$$P(X \leq 5.5) = P\left(Z \leq \frac{5.5 - 5.75}{0.35}\right)$$
$$= P(Z \leq -0.71)$$
$$= \Phi(-0.71) \approx 0.238852$$

c) Find the transfer speed s such that the probability that $X \geq s$ is 90%.
i.e. such that $X \geq s$ 90% of the time.

Want $P(X \geq s) = 0.9$.

i.e. want s such that

$$P(X \le s) = 0.1.$$

equiv:

$$P\left(Z < \frac{s-5.75}{0.35}\right) = 0.1.$$

$$\Phi\left[\frac{s-5.75}{0.35}\right] = 0.1$$

Look up on chart:

$$\Phi(-1.28) = \underbrace{0.100273.}_{\text{close to } 0.1.}$$

So solve

$$-1.28 = \frac{s-5.75}{0.35}$$

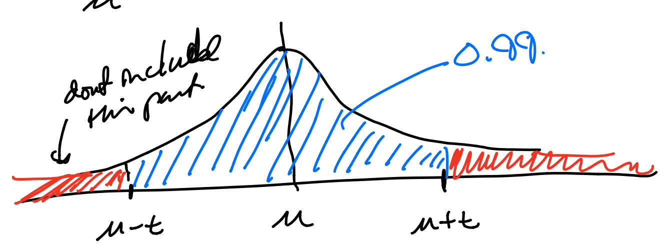
$$s \approx (-1.28)(0.35) + 5.75$$

$$\approx 5.302.$$

d) Find the symmetric interval about the mean such that 99% of the time, the speed is in that interval.

i.e. Find t s.t. $P(\underbrace{5.75}_{\mu} - t \le X \le \underbrace{5.75}_{\mu} + t) = 0.99.$

Draw a picture:



So $\quad 0.99 = P(5.75 - t \le X \le 5.75 + t)$

Observe that

$$1 = P(X \le 5.75 - t) + P(5.75 - t \le X \le 5.75 + t)) + P(X > 5.75 + t)$$

and $P(X \le 5.75 - t) = P(X \ge 5.75 + t)$ by symmetry about $x = 5.75$

So

$$P(5.75 - t \le X \le 5.75 + t) = 1 - 2P(X \le 5.75 - t)$$
$$= 1 - 2P\left(Z \le \frac{\cancel{5.75} - t - \cancel{5.75}}{0.35}\right)$$
$$= 1 - 2P\left(Z \le \frac{-t}{0.35}\right)$$

So if $\quad 0.99 = 1 - 2P\left(Z \le \frac{-t}{0.35}\right)$, then

$$\frac{0.99 - 1}{-2} = 0.005 = \Phi\left[\frac{-t}{0.35}\right].$$

From table $\Phi(-2.57) \approx 0.005087$.

So

$$-2.57 = \frac{-t}{0.35}$$

so $t \approx 0.899 \approx 0.9$

Normal Approx. to Binomial Distribution

- Recall from earlier that the binomial distribution is roughly bell-shaped.

- For large enough n,

$$Bin(n,p) \approx N(np,\ np(1-p)).$$

- It follows that if $X = Bin(n,p)$ then $\frac{X-np}{\sqrt{np(1-p)}}$ is approximated by the standard normal $N(0,1)$.

- For a better approximation, we have a continuity correction: $(X = Bin(n,p))$

$$P(X \le x) = \underbrace{P(X \le x+0.5)}_{\text{equal, since } X \text{ is discrete!!}} \approx \underbrace{P\left(Z \le \frac{x+0.5-np}{\sqrt{np(1-p)}}\right)}_{\Phi\left(\frac{x+0.5-np}{\sqrt{np(1-p)}}\right)}$$

- Similarly:

$$P(X \ge x) = P(X \ge x-0.5) \approx P\left(\frac{x-0.5-np}{\sqrt{np(1-p)}} \le Z\right).$$

- These approximations are good when
$np > 5$ and $n(1-p) > 5$ (large enough n).

<u>Example</u>: multiple choice test, 60 questions with 5 choices each. Guess randomly with equal prob. Find the prob of between 10 and 20 correct.

$p = 1/5$. $1 - p = 4/5$. $n = 60$. $\underbrace{\frac{60}{5} > 5, \ 60\left(\frac{4}{5}\right) > 5,}$
so $Bin(60, 1/5)$ can be approximated well by $N(12, 9.6)$.

$Var(X) = 60\left(\frac{1}{5}\right)\left(\frac{4}{5}\right) \approx 9.6$.
$E(X) = np = 60\left(\frac{1}{5}\right) = 12$

X = # of correct solutions.

$$P(10 \le X \le 20) = \underbrace{\sum_{x=10}^{20} \binom{60}{x} \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{60-x}}_{\text{ugly to compute!}}$$

<u>But</u>: $P(10 \le X \le 20) = P(9.5 \le X \le 20.5)$

$$\approx P\left(\frac{9.5-12}{\sqrt{9.6}} \le Z \le \frac{20.5-12}{\sqrt{9.6}}\right)$$

$$\approx 0.788.$$

$$\sum_{x=10}^{20} \binom{60}{x} \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{60-x} \approx 0.782$$

not to shabby.

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du$$

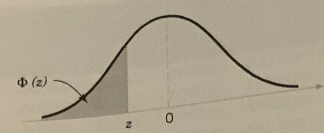


TABLE III Cumulative Standard Normal Distribution

z	−0.09	−0.08	−0.07	−0.06	−0.05	−0.04	−0.03	−0.02	−0.01	−0.00
−3.9	0.000033	0.000034	0.000036	0.000037	0.000039	0.000041	0.000042	0.000044	0.000046	0.000048
−3.8	0.000050	0.000052	0.000054	0.000057	0.000059	0.000062	0.000064	0.000067	0.000069	0.000072
−3.7	0.000075	0.000078	0.000082	0.000085	0.000088	0.000092	0.000096	0.000100	0.000104	0.000108
−3.6	0.000112	0.000117	0.000121	0.000126	0.000131	0.000136	0.000142	0.000147	0.000153	0.000159
−3.5	0.000165	0.000172	0.000179	0.000185	0.000193	0.000200	0.000208	0.000216	0.000224	0.000233
−3.4	0.000242	0.000251	0.000260	0.000270	0.000280	0.000291	0.000302	0.000313	0.000325	0.000337
−3.3	0.000350	0.000362	0.000376	0.000390	0.000404	0.000419	0.000434	0.000450	0.000467	0.000483
−3.2	0.000501	0.000519	0.000538	0.000557	0.000577	0.000598	0.000619	0.000641	0.000664	0.000687
−3.1	0.000711	0.000736	0.000762	0.000789	0.000816	0.000845	0.000874	0.000904	0.000935	0.000968
−3.0	0.001001	0.001035	0.001070	0.001107	0.001144	0.001183	0.001223	0.001264	0.001306	0.001350
−2.9	0.001395	0.001441	0.001489	0.001538	0.001589	0.001641	0.001695	0.001750	0.001807	0.001866
−2.8	0.001926	0.001988	0.002052	0.002118	0.002186	0.002256	0.002327	0.002401	0.002477	0.002555
−2.7	0.002635	0.002718	0.002803	0.002890	0.002980	0.003072	0.003167	0.003264	0.003364	0.003467
−2.6	0.003573	0.003681	0.003793	0.003907	0.004025	0.004145	0.004269	0.004396	0.004527	0.004661
−2.5	0.004799	0.004940	0.005085	0.005234	0.005386	0.005543	0.005703	0.005868	0.006037	0.006210
−2.4	0.006387	0.006569	0.006756	0.006947	0.007143	0.007344	0.007549	0.007760	0.007976	0.008198
−2.3	0.008424	0.008656	0.008894	0.009137	0.009387	0.009642	0.009903	0.010170	0.010444	0.010724
−2.2	0.011011	0.011304	0.011604	0.011911	0.012224	0.012545	0.012874	0.013209	0.013553	0.013903
−2.1	0.014262	0.014629	0.015003	0.015386	0.015778	0.016177	0.016586	0.017003	0.017429	0.017864
−2.0	0.018309	0.018763	0.019226	0.019699	0.020182	0.020675	0.021178	0.021692	0.022216	0.022750
−1.9	0.023295	0.023852	0.024419	0.024998	0.025588	0.026190	0.026803	0.027429	0.028067	0.028717
−1.8	0.029379	0.030054	0.030742	0.031443	0.032157	0.032884	0.033625	0.034379	0.035148	0.035930
−1.7	0.036727	0.037538	0.038364	0.039204	0.040059	0.040929	0.041815	0.042716	0.043633	0.044565
−1.6	0.045514	0.046479	0.047460	0.048457	0.049471	0.050503	0.051551	0.052616	0.053699	0.054799
−1.5	0.055917	0.057053	0.058208	0.059380	0.060571	0.061780	0.063008	0.064256	0.065522	0.066807
−1.4	0.068112	0.069437	0.070781	0.072145	0.073529	0.074934	0.076359	0.077804	0.079270	0.080757
−1.3	0.082264	0.083793	0.085343	0.086915	0.088508	0.090123	0.091759	0.093418	0.095098	0.096801
−1.2	0.098525	0.100273	0.102042	0.103835	0.105650	0.107488	0.109349	0.111233	0.113140	0.115070
−1.1	0.117023	0.119000	0.121001	0.123024	0.125072	0.127143	0.129238	0.131357	0.133500	0.135666
−1.0	0.137857	0.140071	0.142310	0.144572	0.146859	0.149170	0.151505	0.153864	0.156248	0.158655
−0.9	0.161087	0.163543	0.166023	0.168528	0.171056	0.173609	0.176185	0.178786	0.181411	0.184060
−0.8	0.186733	0.189430	0.192150	0.194894	0.197662	0.200454	0.203269	0.206108	0.208970	0.211855
−0.7	0.214764	0.217695	0.220650	0.223627	0.226627	0.229650	0.232695	0.235762	0.238852	0.241964
−0.6	0.245097	0.248252	0.251429	0.254627	0.257846	0.261086	0.264347	0.267629	0.270931	0.274253
−0.5	0.277595	0.280957	0.284339	0.287740	0.291160	0.294599	0.298056	0.301532	0.305026	0.308538
−0.4	0.312067	0.315614	0.319178	0.322758	0.326355	0.329969	0.333598	0.337243	0.340903	0.34457
−0.3	0.348268	0.351973	0.355691	0.359424	0.363169	0.366928	0.370700	0.374484	0.378281	0.38208
−0.2	0.385908	0.389739	0.393580	0.397432	0.401294	0.405165	0.409046	0.412936	0.416834	0.42074
−0.1	0.424655	0.428576	0.432505	0.436441	0.440382	0.444330	0.448283	0.452242	0.456205	0.4601
0.0	0.464144	0.468119	0.472097	0.476078	0.480061	0.484047	0.488033	0.492022	0.496011	0.5000