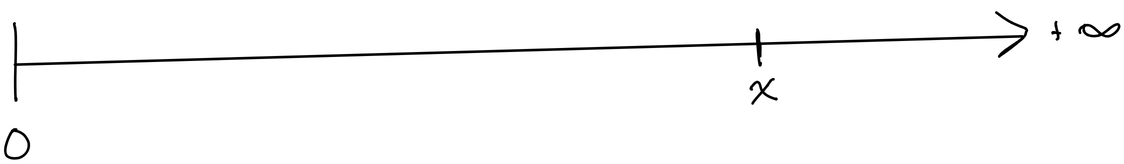
# Lecture 13.

## Exponential distribution.

Consider an interval




Suppose that some event occurs randomly throughout the interval, and occurs on average $\lambda$ times/unit interval, distributed as a Poisson Random Var

Let $X$ be the CRV defined by

$X =$ the length along the interval (from 0, our starting point) until we find an event.

What is $P(X > x)$? $P(X > x)$ is the same as the probability that <u>no</u> events occur in the interval $[0, x]$, and so if

$N_x$ = # of events in $[0, x]$ (and so $N_x$ is Poisson) we have

$$P(X > x) = P(N_x = 0) = \frac{e^{-\lambda x}(\lambda x)^0}{0!}$$
$$= e^{-\lambda x}$$

Thus, $\underbrace{P(X \leq x) = 1 - e^{-\lambda x}}_{\text{cummulative dist. function of } X.}$

<u>Recall</u>: to get the pdf from the cdf, we differentiate:

$$P(X = x) = \lambda e^{-\lambda x} \quad \text{for } 0 \leq x < \infty.$$

<u>Notice</u> that $X$ depends on the length $(x)$ of the interval, not on where the interval starts (starting at $0$ was an arbitrary choice).

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If $X$ is an exponential CRV with parameter $\lambda$, then

$$E[X] = \int_0^\infty x \lambda e^{-\lambda x}\, dx = \lim_{t \to \infty} \int_0^t x \lambda e^{-\lambda x}\, dx = \frac{1}{\lambda}.$$

$$V(X) = E[X^2] - (E[X])^2 = 1/\lambda^2.$$

Ex: The average time between buses at your stop is 15 minutes. If the time between buses is exponentially distributed,

a) What is the probability that you get to the stop and a bus arrives in the next 5 minutes?

b) What is the probability that a bus arrives in the next 5 minutes given that you've already waited 20 minutes?

Let $X$ = waiting time for the bus.

What is $\lambda$? $\quad 15 = \frac{1}{\lambda}$, so $\lambda = \frac{1}{15}$.

So pdf: $\quad f(x) = \frac{e^{-x/15}}{15}$

cdf: $F(x) = P(X \le x) = 1 - e^{-x/15}$

a) $P(X \le 5) = F(5) = 1 - e^{-5/15} = 1 - e^{-1/3}$
$\approx 0.283 \approx 28.3\%$.

b) We use the conditional prob. formula:

$$P(X \le \underbrace{25}_{\text{20 mins + 5 more mins.}} \mid X \geq 20) = \frac{P(20 \le X \le 25)}{P(X \geq 20)}$$

$$= \frac{P(20 \le X \le 25)}{1 - P(X \le 20)} = \frac{F(25) - F(20)}{1 - F(20)}$$

$$= \frac{(\cancel{1} - e^{-25/15}) - (\cancel{1} - e^{-20/15})}{\cancel{1} - (\cancel{1} - e^{-20/15})}$$

$$= \frac{-e^{-25/15} + e^{-20/15}}{e^{-20/15}} = -e^{-5/15} + 1 \approx \underline{\underline{0.283}}$$ $\underline{\underline{\text{The Same}}}$ as part a).

This illustrates what we mean when we say X only depends on the length of the interval (in this case, the waiting time) and not the starting point.

We say that X has the Lack of Memory Property if

$$P(X < t_1 + t_2 \mid X > t_1) = P(X < t_2).$$

(i.e. exponential random vars have this property).

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## $\underline{\text{Chapter 5}}$ Joint Prob. Distributions.

- For this section we only work with continuous distributions.
- much of the material works in the discrete setting; see text.

- Let X, Y be two continuous RVs.
- X and Y may or may not be related.

$\underline{\text{Defn}}$ The "joint probability density function for X and Y is a function $f_{XY}(x,y)$ satisfying the following properties:

1) $f_{XY}(x,y) \geq 0 \quad \forall x,y.$

2) $\iint f_{XY}(x,y)\,dx\,dy = 1.$

3) For any region $R$,

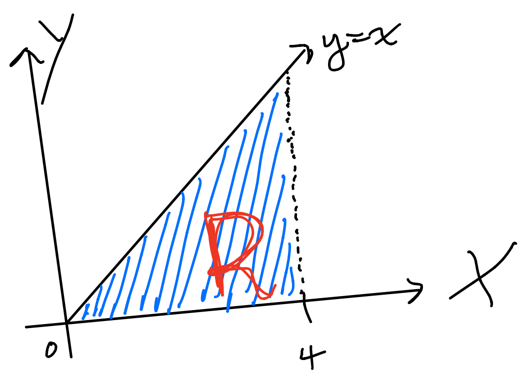
$$P((X,Y) \in R) = \iint_R f_{XY}(x,y)\,dx\,dy.$$

**Ex:** Suppose $X$ and $Y$ are CRV's with $0 < Y < X < 4$ and

$$f_{XY}(x,y) = \underset{\sim}{c}(x+y).$$

$c$ is some constant.

Domain:




Need to find $c$:

$$\underline{1} = \iint_R c(x+y)\,dy\,dx = \int_0^4 \int_0^x c(x+y)\,dy\,dx$$

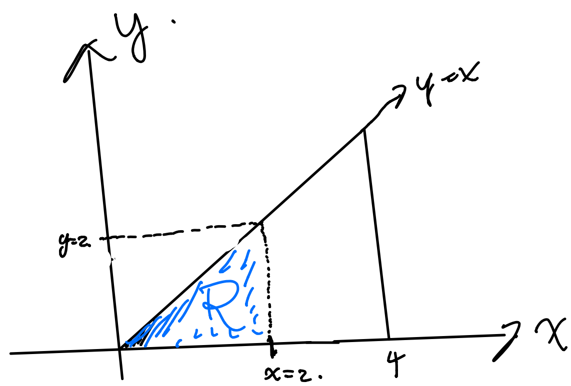
so

$$\underline{1} = c \int_0^4 xy + \frac{y^2}{2} \Big|_0^x dx = \int_0^4 x^2 + \frac{x^2}{2}\,dx$$

$$\underline{1} = c\int_0^4 \frac{3}{2}x^2\,dx = \frac{cx^3}{2}\Big|_0^4 = c\,\frac{2^6}{2} = c\,2^5$$

SO $\quad c = \frac{1}{2^5} = \frac{1}{32}$

Suppose we want $P(X<2,\ Y<3)$
i.e. the prob that $X<2$ AND $Y<3$.



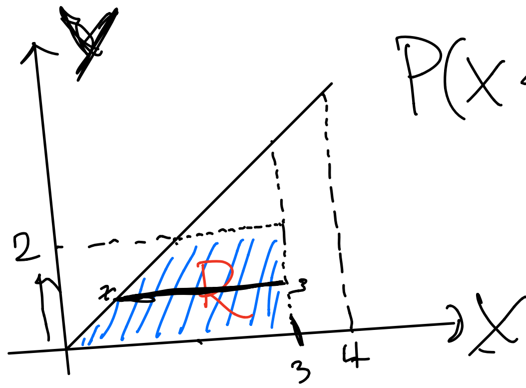

Then

$$P(X<2, Y<3) = \int_0^2\int_0^x \frac{x+y}{32}\,dy\,dx$$

$$= \int_0^2 \frac{3}{64}x^2\,dx = \frac{x^3}{64}\Big|_0^2 = \frac{8}{64} = \frac{1}{8}.$$

Remark: Let $R \subseteq \mathbb{R}^2$ be some 1-dimensional region. Then $P((X,Y)\in R) = 0$.

What is $P(X<3, Y<2)$?




$$P(X<3, Y<2) = \int_0^2 \int_y^3 \frac{x+y}{32}\,dx\,dy.$$

$$= \frac{1}{32}\int_0^2 \frac{x^2}{2} + xy \Big|_y^3 dy.$$

$$= \frac{1}{32}\int_0^2 \frac{9}{2} + 3y - \frac{3}{2}y^2 dy$$

$$= \frac{1}{32}\left(9+6-4\right) = \frac{11}{32}.$$