Lecture 14.

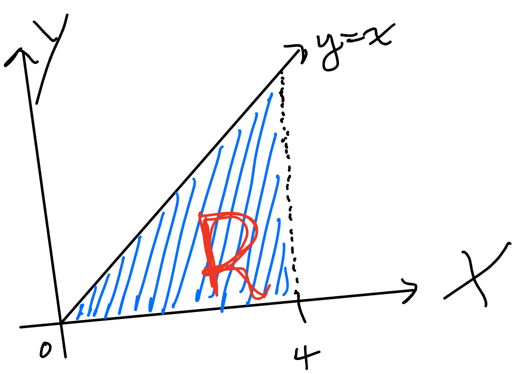
From last time:

Ex: Suppose X and Y are CRV's with $0 < Y < X < 4$ and

$$f_{XY}(x,y) = c(x+y).$$

c is some constant.

Domain:

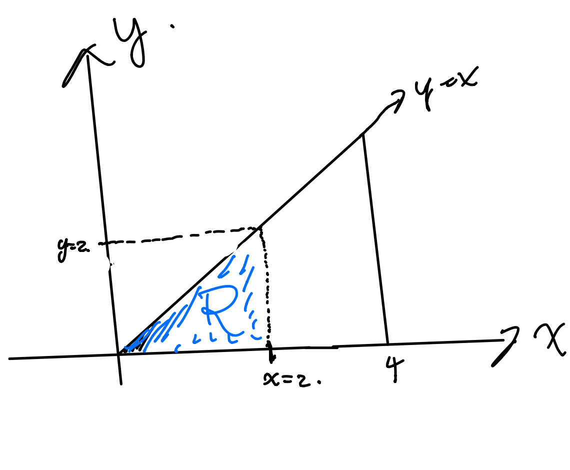


we found that $c = 1/32$,

so $f_{XY}(x,y) = \frac{x+y}{32}$

Suppose we want $P(X<2, Y<3)$

i.e. the prob. that $X<2$ AND $Y<3$.



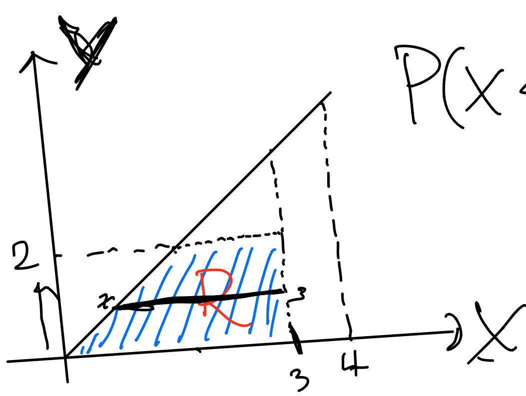
Then

$$P(X<2, Y<3) = \int_0^2 \int_0^x \frac{x+y}{32}\, dy\, dx$$

$$= \int_0^2 \frac{3}{64} x^2\, dx = \frac{x^3}{64}\Big|_0^2 = \frac{8}{64} = \frac{1}{8}.$$

<u>Remark:</u> Let $R \subseteq \mathbb{R}^2$ be some 1-dimensional region. Then $P((X,Y) \in R) = 0$.

What is $P(X<3, Y<2)$?



$$P(X<3, Y<2) = \int_0^2 \int_y^3 \frac{x+y}{32}\, dx\, dy.$$

$$= \frac{1}{32}\int_0^2 \frac{x^2}{2} + xy \Big|_y^3 dy.$$

$$= \frac{1}{32}\int_0^2 \frac{9}{2} + 3y - \frac{3}{2}y^2 dy$$

$$= \frac{1}{32}\left(9+6-4\right) = \frac{11}{32}.$$

Marginal Probability density Functions

Suppose we have CRU's X, Y defined on a region $(X,Y) \in R$, and suppose that X and Y have a joint pdf $f_{XY}(x,y)$,

so $$\iint_R f_{XY}(x,y)\,dx\,dy = 1.$$

In this situation, it is important to know (and be able to obtain) the pdf's of X and Y as CRU's in their own right. We can obtain these functions, the <u>marginal probability density functions</u> as follows:

For a, b let

$$R_X(a) := \{y : (a,y) \in R\}.$$
$$R_Y(b) = \{x : (x,b) \in R.\}.$$

The marginal probability densities are:

$$f_X(x) = \int_{R_X(x)} f_{XY}(a,y)\,dy.$$

$$f_Y(y) = \int_{R_Y(y)} f_{XY}(x,b)\,dx.$$

using these, we can recover $E[X], E[Y], V(X), V(Y)$ in the usual way.

Let's see what happens in our first ex:

$$0 \le Y \le X \le 4, \quad f_{XY}(x,y) = \frac{x+y}{32}.$$

$$f_X(x) = \int_0^x \frac{x+y}{32}\,dy = \frac{1}{32}\left(xy + \frac{y^2}{2}\right)\Big|_0^x = \boxed{\frac{3}{64}x^2}$$

$$f_Y(y) = \int_y^4 \frac{x+y}{32}\,dx = \frac{1}{32}\left(\frac{x^2}{2} + xy\right)\Big|_y^4$$
$$= \frac{1}{32}\left(8 + 4y - \left(\frac{y^2}{2} + y^2\right)\right)$$
$$= \boxed{\frac{1}{32}\left(8 + 4y - \frac{3}{2}y^2\right)}$$

Check: $\int_0^4 f_X(x)\,dx = 1$, $\int_0^4 f_Y(y)\,dy = 1$.

Note: $E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{XY}(x,y)\,dx\,dy.$

and similarly for variance.

Independent Random Variables.

Recall that two events A and B are independent if $P(A \cap B) = P(A)P(B)$.

This motivates a definition for CRVs as well:

Suppose that X and Y are CRV's, jointly distributed with joint prob. density function $f_{XY}(x,y)$. Then:

Def: X and Y are independent if

$$f_{XY}(x,y) = f_X(x) f_Y(y).$$

Fact: If X and Y are independent, then R is rectangular i.e. $R = R_1 \times R_2$ and

$$1 = \iint_R f_{XY}(x,y)\,dx\,dy = \iint_{R_1 \times R_2} f_{XY}(x,y)\,dx\,dy = \left(\int_{R_1} f_X(x)\,dx\right)\left(\int_{R_2} f_Y(y)\,dy\right)$$

For example, $0 \le Y \le X \le 4$, $f_{XY}(x,y) = \frac{x+y}{32}$,

$\frac{(x+y)}{32} \neq f_X(x) f_Y(x)$, and so X and Y are not indep.

This is **not** sufficient. i.e. you can have rectangular domain and still be dependent.

Covariance & Correlation.

Let X and Y be two CRV's with joint probability density function $f_{XY}(x,y)$. If $h(X,Y)$ is a function of X and Y, then the expected value of $h(X,Y)$ is the average weighted by the probabilities:

$$E(h(X,Y)) = \iint h(X,Y) f_{XY}(x,y)\,dx\,dy$$

The covariance of two CRV's gives us a way to measure/detect linear relationships: The covariance is

$$\sigma_{XY} = E[XY] - E[X]E[Y].$$

Notice that if X and Y are independent, then

$$E[XY] = \iint xy f_{XY}(x,y)\,dx\,dy = \iint xy f_X(x) f_Y(y)\,dx\,dy$$
$$= \left(\int x f_X(x)\,dx\right)\left(\int y f_Y(y)\,dy\right)$$
$$= E[X]\,E[Y]$$

and so $\sigma_{XY} = 0$.

However, $\sigma_{XY} = 0$ does not imply that X and Y are unrelated, just that any relation (that they may have) is non-linear.

Next time: Let X be the uniform RV on $[-1,1]$ and $Y = X^2$. Check: $\sigma_{XY} = 0$, but X and Y are not independent!