Lecture 20:

Recall:

- An estimator $\hat{\Theta}$ for θ is unbiased if
  $E[\hat{\Theta}] = \theta$.

- We saw that the sample mean, $\overline{X}$ is an unbiased estimator.

- However, $\overline{X}$ is not the only unbiased estimator of the mean.

- For example: the following are unbiased estimators for μ:
  - the median of a random sample
  - the 10% trimmed mean:

    $\overline{X}_{tr(10)}$ = the average not including the top 10% and bottom 10% of the data.

- How do we decide what estimator to use?

- Since an estimator $\hat{\Theta}$ is a RV, the variance is a measure of how spread out the sampling distribution of $\hat{\Theta}$ is.

- Therefore, if $\hat{\Theta}_1$ and $\hat{\Theta}_2$ are two unbiased estimators for θ, and $V(\hat{\Theta}_1) \leq V(\hat{\Theta}_2)$, then point estimate of θ by $\hat{\Theta}_1$ will, on average, be more likely to be close to θ, than point estimate for θ via $\hat{\Theta}_2$.

- This suggests a common method for choosing an estimator: minimize the variance.

<u>Defn</u>: Let θ be a parameter. Of all estimators for θ, the unbiased estimator $\hat{\Theta}$ with minimal variance is called the <u>minimum variance unbiased estimator</u> (MVUE).

<u>Example</u>: If X is a RV with mean μ and variance σ^2, and $\{X_1, \ldots, X_n\}$ is a random sample of X, then X_1 and $\overline{X}$ are both unbiased estimators for μ, but $V(X_1) = \sigma^2$ and $V(\overline{X}) = \frac{\sigma^2}{n}$.

- There are general techniques for determining MVUE's for parameters but these are beyond the scope of the course.

- One important fact:

Suppose that $X_1, \ldots, X_n$ is a random sample from a normal distribution with mean μ and variance σ^2. Then $\overline{X}$ is the MVUE for μ.

- Another way we can choose good estimators is by minimizing the error.

- Error is a measurement of how precise an estimator is.

- Given an estimator $\hat{\Theta}$, the standard error of $\hat{\Theta}$ is $\sqrt{V(\hat{\Theta})} = \sigma_{\hat{\Theta}}$.

- One issue with this is that the standard error may contain unknown parameters

- if these parameters can be estimated, we can substitute them in to get the estimated standard error, denoted by $S_{\hat{\Theta}}$, $SE(\hat{\Theta})$, or $\hat{\sigma}_{\hat{\Theta}}$

Ex: Suppose that we are sampling $N(\mu, \sigma^2) = X$ with a sample size n. The sample mean $\overline{X}$ is an unbiased estimator of μ, and $\overline{X}$ is again normal with mean μ and variance σ^2/n. Therefore, the standard error of $\overline{X}$ is

$$\sigma_{\overline{X}} = \sqrt{V(\overline{X})} = \frac{\sigma}{\sqrt{n}}.$$

Now, if σ is also an unknown parameter we can substitute an estimator for σ to get the/a estimated standard Error of $\sigma_{\overline{X}}$;

$$\hat{\sigma}_{\overline{X}} = \frac{S}{\sqrt{n}}. \quad \text{(if } S \text{ is an estimator for } \sigma\text{)}.$$

Biased Estimators.

- Sometimes, we have no choice but to a biased estimator (for exc: often use S as an estimator for σ).
- when using biased estimators, it is often useful to consider the mean square error. (MSE).
- If $\hat{\Theta}$ is an estimator for θ, then we define:

$$MSE(\hat{\Theta}) := \underbrace{E[(\hat{\Theta} - \theta)^2]}_{\text{second moment of } \hat{\Theta} - \theta.}$$

Note that we can write.

$$MSE(\hat{\Theta}) = E\left([\hat{\Theta} - E[\hat{\Theta}]]^2\right) + (\theta - E[\hat{\Theta}])^2$$

$$= V(\hat{\Theta}) + (-(E[\hat{\Theta}] - \theta))^2$$

$$= V(\hat{\Theta}) + (\underbrace{E[\hat{\Theta}] - \theta}_{\text{bias}})^2$$

Notice that this generalizes the (square of) the standard error in the case of unbiased estimators, since if $E[\hat{\Theta}] - \theta = 0$, then

$$MSE(\hat{\Theta}) = V(\hat{\Theta}) = \sigma_{\hat{\Theta}}^2.$$

- The MSE gives a good way of comparing two estimators.
- If θ is a parameter and $\hat{\Theta}_1$ and $\hat{\Theta}_2$ are estimators, then the relative efficiency of $\hat{\Theta}_2$ to $\hat{\Theta}_1$ is defined to be

$$\frac{MSE(\hat{\Theta}_1)}{MSE(\hat{\Theta}_2)}.$$

If the rel. efficiency is < 1 we say that $\hat{\Theta}_1$ is a more efficient estimator of θ than $\hat{\Theta}_2$, in the sense that it has smaller MSE.

An estimator $\hat{\Theta}$ for θ such that

$$\frac{MSE(\hat{\Theta})}{MSE(\Theta')} < 1$$

for any θ-estimator $\Theta' \neq \hat{\Theta}$ is called optimal,

but such estimators rarely exist.