Lecture 24.

Now, suppose we want a $100(1-\alpha)\%$ CI for the proportion with a certain amount of error, $E = z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$.

- what should n be?

$$n = \left\lceil \frac{z_{\alpha/2}^2}{E^2} \underbrace{(\hat{p}(1-\hat{p}))} \right\rceil$$

but what if we don't know $\hat{p}$??

2 methods:

① use an earlier study and compute $\hat{p}$ from that (eg: in the last example, found $\hat{p} = 0.36$; use that).

② Observe that $p(1-p) \leq 0.25$ (when $p=0.5$) and so substitute $\hat{p} = 0.5$ as a worst case scenario: $n = \left\lceil \left(\frac{z_{\alpha/2}}{E}\right)^2 (0.25) \right\rceil$

Ex: We want a 95% CI for the proportion of eng students dressing up for halloween with error at most $E = 0.07$. What should the sample size be?

$$\alpha = 0.05 \longrightarrow z_{\alpha/2} = 1.96.$$

So $n = \left\lceil \left(\frac{1.96}{0.07}\right)^2 (\hat{p}(1-\hat{p})) \right\rceil$.

method ①: In the small test study (sample size 50) we had $\hat{p} = 0.36$.

So $n \geq \left\lceil \left(\frac{1.96}{0.07}\right)^2 (0.36)(1-0.36) \right\rceil = \lceil 180.36 \rceil = 181$

method ②: (worst case scenario) $\hat{p}(1-\hat{p}) = 0.25$

So $n \geq \left\lceil \left(\frac{1.96}{0.07}\right)^2 (0.25) \right\rceil = 196.$

Hypothesis Testing

(for a single sample)

Chapter 9.

- In the last sections we were concerned with estimating some parameter associated to a population
  ↳ estimate with an estimator or confidence interval.

- In this chapter we aim to test whether a statistical hypothesis is true

- For the purposes of this course, a <u>statistical hypothesis</u> will be a statement about the parameters of one or more populations.

- The setup for hypothesis testing is usually as follows:

H_0 (Null Hypothesis): the statement we assume initially to be true

↳ Eg: $\mu = 50$

H_1 (Alternate hypothesis): A statement that contradicts the null Hypothesis.

↳ eg: $\mu \neq 50$ (two-sided)

eg: $\mu \gtrless 50$ (one-sided)

eg $\mu \lessgtr 50$ (one-sided)

Strategy:

- Assume H_0.
- Consider the data: is there something that is very unlikely/implausible if H_0 is true?
- If Yes, reject H_0 in favour of H_1.
- If no, accept H_0

Ex: A water bottle factory tells you the volume of water bottles is normally distributed with a mean $\mu = 100$ml and $\sigma = 5$ml. Let H_0 ~~be~~ "$\mu = 100$ml", H_1: $\mu \neq 100$ml. You take a sample of 1000 bottles and find that the sample mean is 90 ml. Should you accept H_0?

Objectives of Hypothesis Testing:

1) Test if a parameter has changed (last ex).
2) Test a theory (i.e. H_0 is a theoretical assumption)
3) Conformance testing (H_0: is the parameter what it should be?).

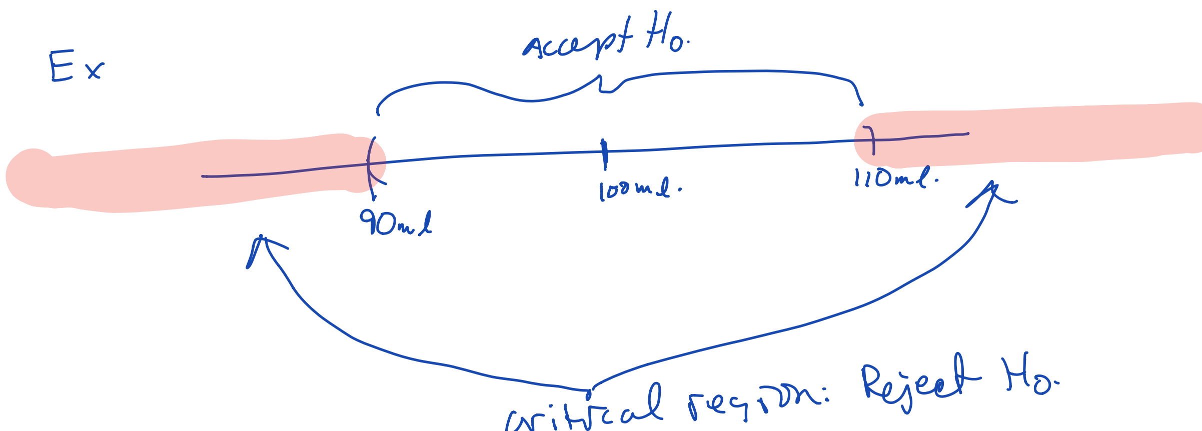
Method:

① Assume H_0 and consider how likely a "test statistic" takes a given value. Ex: $\bar{x}$, s^2, etc.

② Pick a test statistic (eg an estimator)
↳ eg. $H_0: \mu = 100\,ml$. Then can choose $\bar{X}$ as a test statistic.

③ Specify a critical region (determined by critical values)
↳ i.e. value of the test statistic that result in rejecting H_0.

Ex



④ Check if test statistic is in the critical region or not.

Key: Should determine the critical region based on how likely the sample statistic is to take a value there if H_0 is true.

Ex: In the waterbottle example, we have a sample size $n = 1000$, so $\overline{X}$, the sample mean is $N(100, \frac{25}{1000})$

So $P(90 < \overline{X} < 110) = P(-63.25 < Z < 63.25) \sim 1$ (essentially guaranteed.)

So if we actually find a sample of size 1000 with average volume ≤ 90 or ≥ 110, we should reject $H_0: \mu = 100$. (1000 here is a gross overestimate. Even with a sample size of 10 you should reject H_0).

Possible Outcomes

	H_0 True	H_1 True.
Accept H_0	✓	Type II error
Reject H_0	Type I error	✓

Define: $\alpha = P(\text{Type I error}) :=$ "significance level" of the test.

$\beta = P(\text{Type II error})$, $1-\beta$ is the "power" of the test.

$1-\beta =$ prob. of correctly rejecting H_0.

Example: Suppose we sample a normal distribution with $\sigma = 10$, with sample size 35.

Suppose we have
$$H_0 : \mu = 50$$
$$H_1 : \mu \neq 50$$

Suppose our critical region is $\bar{x} \geq 52$, $\bar{x} \leq 48$.

a) Find α.

b) Suppose $\mu = 53$. What is β?

a) $\alpha = P(\text{Reject } H_0, H_0 \text{ true}) = P(\bar{X} \geq 52 \text{ or } \bar{X} \leq 48 \text{ and } \mu = 50)$

$= P(\bar{X} \geq 52, \mu = 50) + P(\bar{X} \leq 48, \mu = 50)$

$= \underbrace{P\left(Z \geq \frac{52-50}{10/\sqrt{35}}\right) + P\left(Z \leq \frac{48-50}{10/\sqrt{35}}\right)}_{Z \text{ scores.}} = 0.238.$

b) $\beta = P(\text{accept } H_0 \text{ given that } H_0 \text{ is false})$

if $\mu = 53$, then

$\beta = P(\text{accept } H_0 \text{ given that } \mu = 53)$

$= P\left(\frac{48-53}{10/\sqrt{35}} \leq Z \leq \frac{52-53}{10/\sqrt{35}}\right) = 0.274.$