Lecture 27.

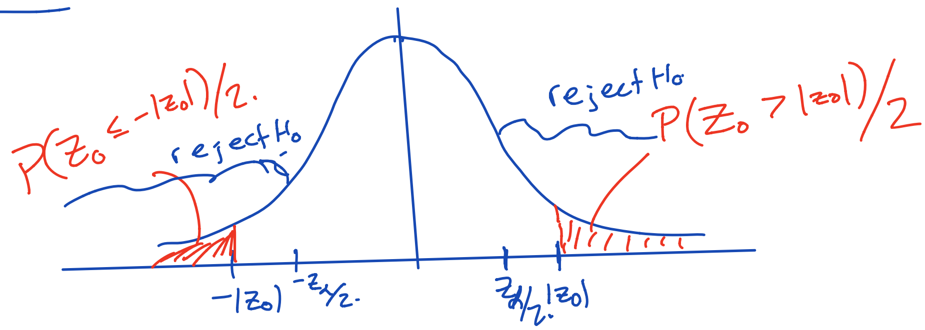
Z-test Structure (for fixed sig. level α).

i) From the data, compute $z_0 = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$ } some #.

ii) Find the p-value for the test with observed data z_0.

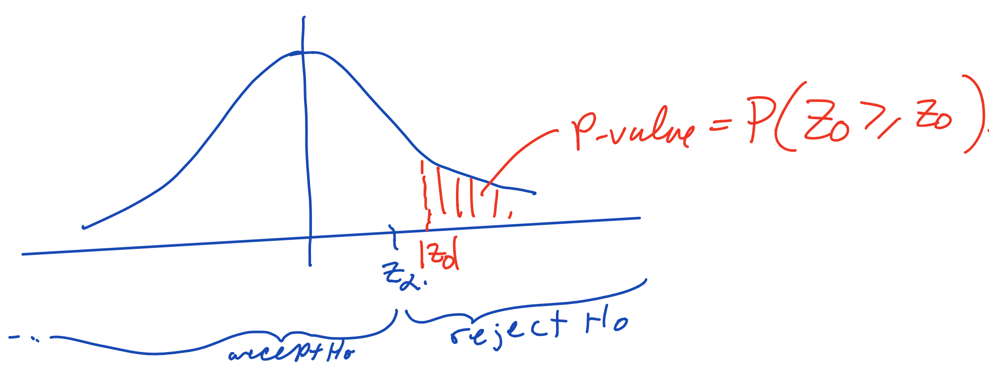
iii) Reject H_0 if $\underbrace{\text{p-value of } z_0 \text{ is } \leq \alpha}_{\text{what does this mean? cases.}}$.

Case I: $H_1 : \mu \neq \mu_0$.



Reject H_0 iff $|z_0| \geq z_{\alpha/2}$

Case II: $H_1 : \mu \gtrless \mu_0$ (one sided/tailed).



Then reject H_0 if $P(Z_0 \geq z_0) \leq \alpha$.

$$\Updownarrow$$

$$z_0 \geq z_\alpha.$$

where z_α is such that
$$\Phi(z_\alpha) = 1-\alpha.$$

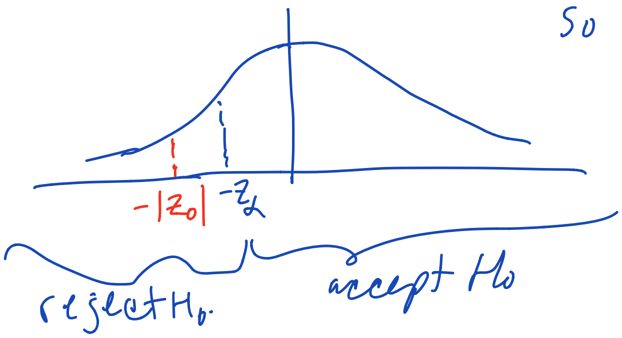
Similarly:

Case III: $\quad H_1: \mu \lneq \mu_0$. (one sided/tailed)

p-value is $P(Z_0 \leq z_0)$

so Reject H_0 if.

$$P(Z_0 \leq z_0) < \alpha.$$
$$\Updownarrow$$
$$z_0 \leq -z_\alpha.$$



Example: Sample of 25 adults have a mean body temperature of 36.8°C. Assume body temp is normally distributed with $\sigma = 0.34$°C. Suppose our test is

$$H_0: \mu = 37°C.$$
$$H_1: \mu \neq 37°C. \leftarrow \text{(two-sided!)} \text{ case I}$$
$$\alpha = 0.01.$$

Accept or Reject H_0?

Solution:
$$z_0 = \frac{\bar{x}-\mu_0}{(\sigma/\sqrt{n})} = \frac{36.8-37}{0.34/\sqrt{25}} = -2.94$$

... p-value (2-sided) of the test is

So the p-value is

$$2P(Z \le -2.94) = 0.003282 \le 0.01 = \alpha$$

So reject H_0!

Type II Error and Choice of Sample Size

- Recall that the result of a hypothesis test is a "Type II error" if we accept H_0 even when it is actually false.
- The "power" of a test is $1-\beta$, where $\beta = P(\text{Type II error})$.
- In a test, the tester preselects $\alpha = P(\text{Type I error})$; the power depends on the sample size.
- In this section, we discuss regulating β based on sample size, assuming we are sampling a normal distribution with known variance.

- Consider the two-sided hypothesis:

$$H_0: \mu = \mu_0 \qquad H_1: \mu \ne \mu_0$$

- Our test statistic is $Z_0 = (\bar{X} - \mu_0)/(\sigma/\sqrt{n})$
- Suppose that H_0 is false: say that $\mu = \mu_0 + \delta$, for some $\delta > 0$. Then

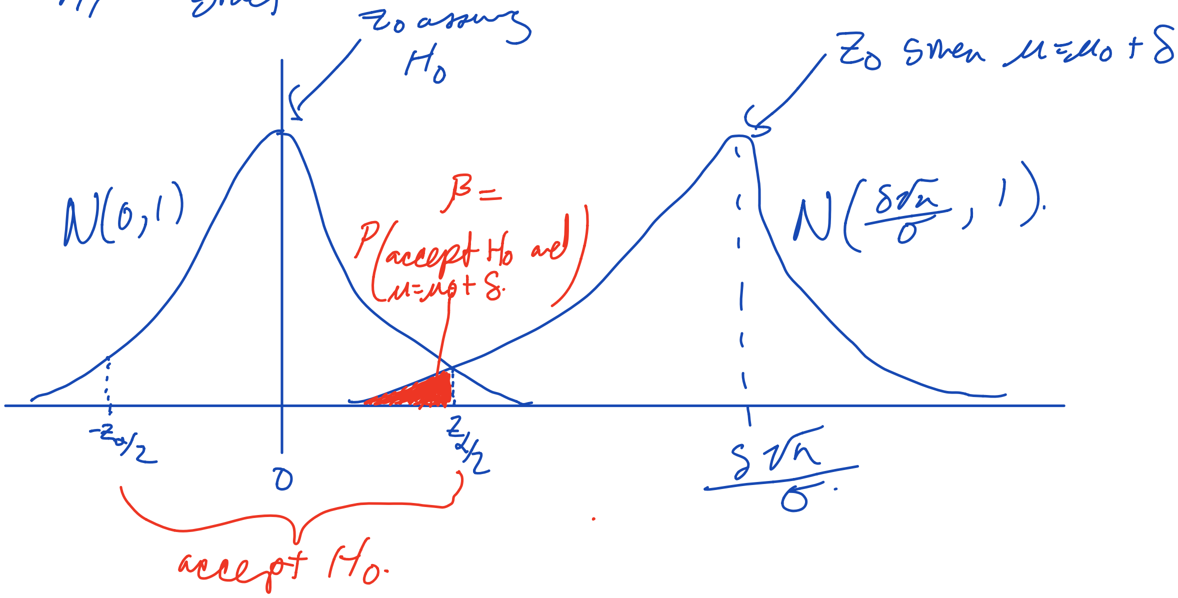
$$E[Z_0] = \frac{E[\bar{X}] - \mu_0}{\sigma/\sqrt{n}} = \frac{(\mu_0 + \delta) - \mu_0}{\sigma/\sqrt{n}} = \frac{\sqrt{n}\,\delta}{\sigma}$$

- Thus under the assumption that H_1 is true (i.e. $\mu = \mu_0 + \delta$) we have

$$Z_0 \sim N\left(\frac{\delta\sqrt{n}}{\sigma}, 1\right).$$

- Plotting Z_0 under the assumption of H_0 and

H_1 given.



So $\quad \beta = P\left(-z_{\alpha/2} \le N\left(\frac{\delta\sqrt{n}}{\sigma}, 1\right) \le z_{\alpha/2}\right)$

$$= \Phi\left(z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right) - \Phi\left(-z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right).$$

Note: - By symmetry this formula still holds when $\delta < 0$
- can derive a similar formula for a one sided hypothesis

Choosing sample size

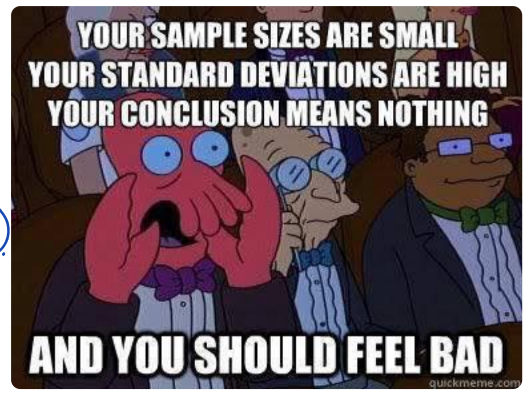
- Suppose $\delta > 0$. Then $\Phi\left(-z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right)$ is v small.
  ↳ so $\quad \beta \approx \Phi\left(z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right)$.
- Let z_β be such that $\Phi(z_\beta) = 1-\beta$ (and so $\Phi(-z_\beta) = \beta$)
- Hence $\qquad -z_\beta \approx z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}$.
- Solving for n, we get

$$n \approx \frac{(z_{\alpha/2} + z_\beta)^2 \sigma^2}{\delta^2}, \quad \delta = \mu - \mu_0.$$

- round up if not an integer.
- Note that small β requires a larger sample size (since β small $\Rightarrow$ $1-\beta$ large $\Rightarrow$ z_β large).
- i.e. "With great power comes great sample size."
- A <u>one-sided</u> test gives a similar formula:

$$n \approx \frac{(z_\alpha + z_\beta)^2 \sigma^2}{\delta^2}, \quad \delta = \mu - \mu_0.$$

(don't mix the two up!).



<u>Ex:</u> Suppose batteries have lifetime in hours $\sim N(\mu, 1.5)$. If the assumption is that $\mu = 40$, but in reality $\mu = 43$, how big does the sample size need to be to ensure that $\beta < 0.05$ in a two-sided test with $\alpha = 0.025$?

<u>Solution:</u> $$n \geq \left\lceil \frac{(z_{\alpha/2} + z_\beta)^2}{\delta^2} \sigma^2 \right\rceil$$

$$= \left\lceil \frac{(z_{0.0125} + z_{0.05})^2 (1.5)}{(43-40)^2} \right\rceil = \left\lceil \frac{(2.24 + 1.64)^2 (1.5)}{9} \right\rceil$$

$$= \lceil 2.51 \rceil = 3.$$

One thing to note: the approximation is only really good when $\Phi\left(-z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right)$ is small compared to β.

Large Sample Sizes

- By the central limit theorem, if is large (usually $n \geq 30$) then we can drop the assumption that we are sampling a normal distribution (since $\bar{X}$ will be approximately normal anyway.

- If the variance is unknown, but the sample size is large enough, say $n \geq 40$, then we may substitute s^2 (sample variance) for σ^2 in all of the above without changing much.