Lecture 28

Tests on the mean of a normal distr. w/ known variance: $Z_0 = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$.

Large sample sizes

- By the central limit theorem, if is large (usually $n \geq 30$) then we can drop the assumption that we are sampling a normal distribution (since $\bar{X}$ will be approximately normal anyway.
- If the variance is unknown, but the sample size is large enough, say $n \geq 40$, then we may substitute s^2 (sample variance) for σ^2 in all of the above without changing much.

a really s

Hypothesis Testing on the mean of a normal distribution, unknown variance, small sample size.

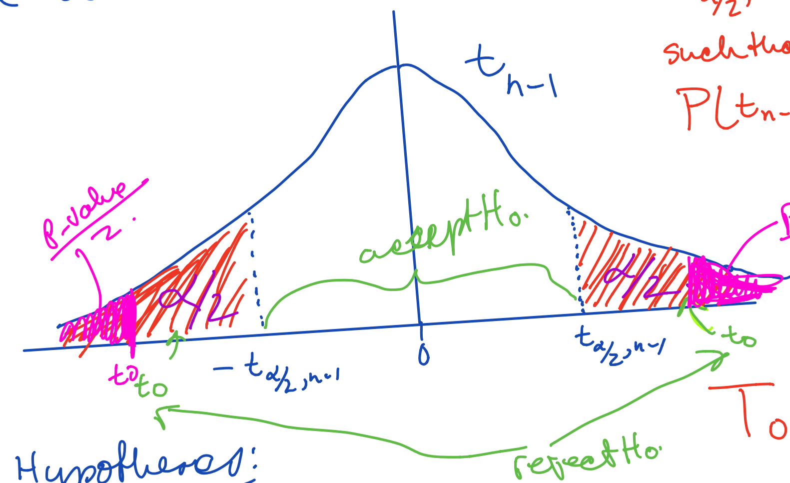
- In this situation we take an approach analogous to what we did for confidence intervals.
- Consider a null hypothesis $\boxed{H_0 : \mu = \mu_0.}$
- We choose as a test statistic

$$T_0 = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$$

where n is the sample size, and s is the sample standard deviation.

- If H_0 is true, then $T_0 = t_{n-1}$, i.e. T_0 is has a Student's t-distribution with $n-1$ degrees of freedom (so these tests are called "t-tests"!)
- Choose α a significance level.

$t_{\alpha/2, n-1}$ is the value such that $P(t_{n-1} < t_{\alpha/2,n-1}) = 1-\frac{\alpha}{2}$.



$$T_0 = \frac{\overline{X} - \mu_0}{s/\sqrt{n}}$$

Alternate Hypotheses:

(I) $H_1: \mu \neq \mu_0$ (two-sided).

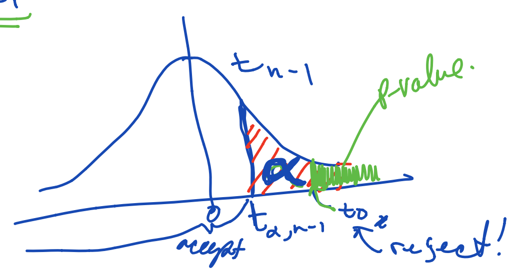
Critical region: $|t_0| > t_{\alpha/2, n-1}$ ← reject H_0!

P-value: $2P(T_0 > |t_0|)$

(II) $H_1: \mu > \mu_0$

Critical region: $t_0 > t_{\alpha, n-1}$ (not $\alpha/2$.) ← reject H_0!

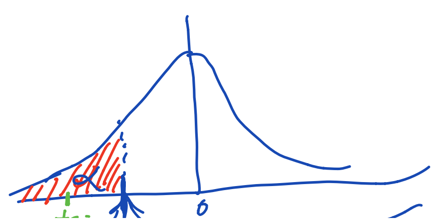
P-value: $P(T_0 > t_0)$



(III) $H_1: \mu < \mu_0$.

Critical region: $t_0 < -t_{\alpha, n-1}$

P-value: $P(T_0 < t_0)$



reject if t_0 is here. $-t_{\alpha, n-1}$. accept if t_0 is here

One thing to note!

↳ when performing t-tests, statistical software can give P-values precisely.

↳ when doing t-tests by hand, this is almost impossible, since the t-table are too coarse.

<u>Ex:</u> consider the following situation:

a) $H_0: \mu = \mu_0$.
$H_1: \mu \neq \mu_0$.
$t_0 = 2.537$
$n = 10$.

b) $H_0: \mu = \mu_0$
$H_1: \mu > \mu_0$.
$t_0 = 1.863$
$n = 16$.

Let's estimate the P-values in these two situations:

a) $n = 10, \to 10 - 1 = 9$ degrees of freedom: (2-sided case).

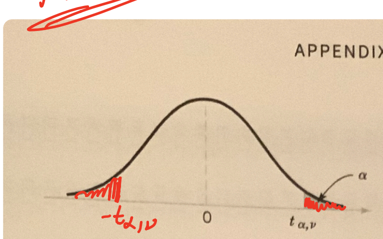
TABLE V Percentage Points $t_{\alpha,\nu}$ of the t Distribution

here $\alpha = \frac{\text{p-value}}{2}$.

ν \ α	.40	.25	.10	.05	.025	.01	.005	.0025	.001	.0005
1	.325	1.000	3.078	6.314	12.706	31.821	63.657	127.32	318.31	636.62
9	.261	.703	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781

degrees of freedom

$t_0 = 2.537$. $\Rightarrow \frac{\text{P-value}}{2}$ is between 0.01 and 0.025



$-t_{\alpha/2,\nu}$

$t_0 = t_{\alpha/2,\nu}$.
↓
find a range for $\alpha/2$.

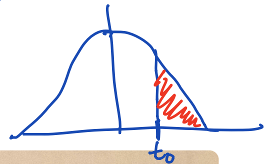
<u>So</u> $0.02 \leq$ P-value ≤ 0.05

i.e $\underbrace{0.02}_{2(0.01)} \leq$ P-value $\leq \underbrace{0.05}_{2(0.025)}$

b) $n = 16$, so 15 degrees of freedom.

$t_0 = 1.863$. (1 sided case)

$H_1: \mu > \mu_0$.



p-value in this range.

α = p-value

TABLE V Percentage Points $t_{\alpha,\nu}$ of the t Distribution

ν \ α	.40	.25	.10	.05	.025	.01	.005	.0025	.001	.0005
1	.325	1.000	3.078	6.314	12.706	31.821	63.657	127.32	318.31	636.62
15	.258	.691	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073

degrees of freedom

$t_0 = 1.863$ in this range.

So p-value is between 0.025 and 0.05.

Tests on Population proportion

- Suppose some population has some particular subclass which is proportion p of the total population:
  - eg: of all engineering students what proportion are female?

- Suppose we take a sample of size n, and let X = # of samples from the subclass.
- Then $X = Bin(n, p)$ and $\hat{p} = \frac{X}{n}$ is an unbiased estimator of p (see notes on CI's for pop. proportion).

- If $np > 5$, $n(1-p) > 5$, then we may further approximate $X \approx N(np, np(1-p))$.

Suppose our test is:

$H_0: p = p_0$

H_1: (I) $p \neq p_0$, (II) $p > p_0$ (III) $p < p_0$.

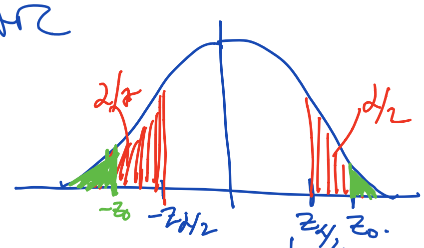
significance level α.

Sub. $p = p_0$

- If H_0 is true, then $X \approx \underline{N(np_0, np_0(1-p_0))}$ and so $\hat{p} = \frac{X}{n} \approx N\left(p_0, \frac{p_0(1-p_0)}{n}\right)$

- We choose our test statistic

$$Z_0 = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$



- Then $Z_0 \approx N(0,1)$ iff H_0 is true!

- Suppose, for some sample, $\underline{Z_0 = z_0}$. Then: ← point estimate.

(I) $H_1: p \neq p_0$ (two-sided). p-value: $2P(Z_0 \geq |z_0|) = 2\Phi(-|z_0|)$ Reject H_0 if $|z_0| \geq z_{\alpha/2}$.

(II) $H_1: p > p_0$ p-value: $P(Z_0 > z_0) = \Phi(-z_0)$ Reject H_0 iff $z_0 \geq z_\alpha$.

(III) $\boxed{H_1: p < p_0}$ p-value: $P(Z_0 < z_0) = \Phi(z_0)$ Reject H_0 iff $z_0 \leq -z_\alpha$.

Example: In 2018, McMaster claimed that 27% of all incoming engineering students were female. Of a random sample of 25 engineering students, 4 are female. At a significance level of $\alpha = 0.01$, does the evidence support McMaster's claim?

Solution: $\hat{p} = \frac{4}{25} = 0.16$

$H_0: p = 0.27$

$H_1: p < 0.27$

$$Z_0 = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{\overset{\hat{p}}{0.16} - \overset{p_0}{0.27}}{\sqrt{\frac{\underset{p_0}{(0.27)}\underset{1-p_0}{(0.73)}}{\boxed{25}}}} = \boxed{-1.2389}$$

For $\alpha = 0.01$, $z_\alpha = 2.33$ from Z-chart.

z_α is the value such that $P(Z < z_\alpha) = 1-\alpha$.

so $-z_\alpha = -2.33$

$z_0 \geq -z_\alpha$

$-1.2389 \geq -2.33$

So don't reject H_0. ✓