Lecture #2.

Recall that an event E is a subset of a sample space S ($E \subseteq S$).

Two events are mutually exclusive if $E_1 \cap E_2 = \emptyset$.

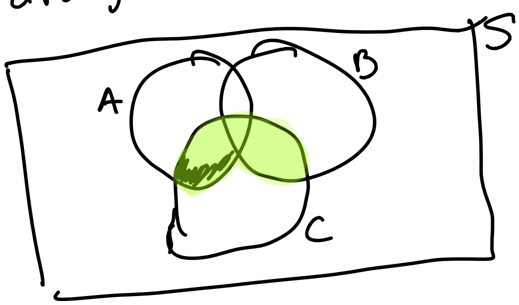
We also have the following laws concerning the operations of $\cup, \cap, '$:

1) $(E')' = E$.

(Dist) 2) i) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
ii) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

DeMorgan's Laws 3) i) $(A \cup B)' = A' \cap B'$
$(A \cap B)' = A' \cup B'$.

Many of these can be visualized with Venn diagrams:



The shaded area is $(A \cup B) \cap C$.

Counting Techniques

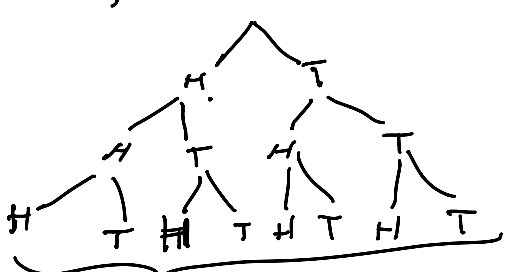
In many situations (especially when we want to compute probabilities) we will want to know how many possible outcomes an experiment may have. Counting techniques (aka combinatorial analysis) often can give effective ways of computing ~~these~~ these things.

Basic Counting Principle

Suppose we have r-many experiments, and the ith experiment has n_i-many possible outcomes. Then the total number of possible outcomes (for experiments run consecutively) is given by the product

$$\prod_{i=1}^{r} n_i = n_1 n_2 \cdots n_r$$

For example, last time we drew a tree diagram for the experiment consisting of three (3) consecutive coin tosses:



Since each flip has $2 \times 2 \times 2 = 8$ outcomes.

Another Example: A "game" of OLG's Lotto Max consists of choosing 7 numbers, each from 1–50.

How many possible games are there?

$$50 \times 50 \times \dots \times 50 = 50^7 \approx 78\underline{1} \text{ billion}$$

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We can now leverage this basic principle to count many other phenomena.

Permutations

How many ways can we arrange the letters a, b, c? Each arrangement is called a permutation. We can write them down

$$\underbrace{abc,\ acb,\ bac,\ bca,\ cab,\ cba}_{6 \text{ ways}}$$

How did we compute this?

$$\underset{\text{3 choices}}{\_\_\_}\quad \underset{\text{2 choices}}{\_\_\_}\quad \underset{\text{one choice}}{\_\_\_} \ : \ 3 \cdot 2 \cdot \underline{1} = 6$$

In general, given n objects, there are

$$n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 \text{ many ways to permute them.}$$

More generally we can count the number of ways to permute r objects chosen out of a pool of n objects.

Ex: How may 3 letter words are there with no repeated letters?

$\underline{\qquad}$ 26 options $\underline{\qquad}$ 25 option $\underline{\qquad}$ 24 options.

Total $= 26 \cdot 25 \cdot 24$.

In general, the same technique gives that the number of ways to permute r elements from a group of n elements is

$$P_r^n = {}_nP_r = n \times (n-1) \times \dots \times (n-r+1) = \frac{n!}{(n-r)!}$$

Permutations with identical Object.

How many unique permutations are there of the letters

banana?

The problem is that some permutations give identical results, i.e. swapping the n's or a's.

If we label the letters so that everything is distinguishable

$b_1 a_2 n_3 a_4 n_5 a_6$

then there are $6!$ many permutations.

Which permutations give the same result?

If we fix $b_1\ a_2\ a_4\ a_6$, and allow ourselves to swap the n's then there are $2!$ many such permutations.

Similarly, if we fix the $b\ n_3, n_5$, we are permuting $a_2\ a_4\ a_6$, and so there are $3!$ permutations.

Therefore there are $2!\,3!$ permutations that leave things unchanged.

It follows that there are

$$\frac{6!}{2!\,3!} \text{ many unique permutations}$$

$$= \frac{6\cdot 5\cdot \cancel{4}^{2}\cdot \cancel{3}\cdot \cancel{2}\cdot 1}{\cancel{2}\cdot \cancel{3}\cdot \cancel{2}\cdot 1} = 60$$

In general, the number of permutations of $n = n_1 + \cdots + n_r$ many objects with n_i identical objects for each i, is

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}$$

Combinations

How many ways can we choose r elements from an n element set?

These are called "combinations".

For example, out of a group of 10 people, how can I choose 5 of them?

Well for the first slot, we have 10 choices

2^{nd} " 9

$\vdots$

5 6 choices.

So $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$. but(!!) the order doesn't matter, so we should "cancel" out any permutations of the group of 5, and we know there are $5!$ such permutations.

So there are

$$\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5!} = \frac{10!}{5!\,5!} \text{ many choices.}$$

In general, the number of subsets of size r chosen from a set of size n is

$$\frac{n!}{(n-r)!\,r!} := \binom{n}{r} = {}_nC_r = C^n_r$$

The numbers $\binom{n}{r}$ are called "binomial coefficients", since we have

The Binomial Theorem

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

Application: Given a set of size n, how many subsets are there?

↳ $\binom{n}{r}$ is the number of subsets of size r, and so the total # of subsets is

$$\sum_{r=0}^{n} \binom{n}{r} = \sum_{r=0}^{n} \binom{n}{r} \underline{1^r} \, \underline{1^{n-r}}$$

$$= \underbrace{(1+1)^n}_{\text{by the binomial theorem.}} = 2^n$$