Conditional Probability.

Last time, we considered the following situation:

- you have a deck of cards
- your friend draws a card
- if you draw a card after your friend, then the probability of you getting an ace depends / is conditional upon the outcome of their draw.
- If they drew an ace then you have a $3/51 \times 100\%$. chance of drawing an ace.
- Otherwise, you have a $4/51$ chance.
- The idea is that given two events, say A and B, the probability that event A occurs may depend on whether B has occurred.
- we write $P(A|B)$ to mean "the probability of A assuming that B has occured".

We have a nice formula for conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Ex: Imagine the experiment where we roll one die and then another. We know that the sample space has size $6 \times 6 = 36$.

Question: What is the probability that the total of the two rolls is greater than 7 given that the first roll is a three?

A = "the total is greater than 7"

B = "first roll is 3".

First of all, what is $P(B)$? When rolling a die, each outcome is ~~equally~~ likely. Since there is only one "3", $P(B) = 1/6$.

Now, what is $P(A \cap B)$?

Recall that the intersection is

$A \cap B$ = "the total is greater than 7 <u>and</u> the first roll is 3"

We know that if n is the second roll, then

$n + 3 > 7$ iff $n > 7 - 3 = 4$.

So $P(A \cap B) = P($"the second roll is greater than 4"$)$

Since there are only two outcomes of a single die roll greater than 4 $(5, 6)$ we have $P(A \cap B) = 2/36 = 1/18$.

So $P(A|B) = \dfrac{1/18}{1/6} = \dfrac{6}{18} = \dfrac{2}{6} = 1/3 = 33.333\ldots\%$

Remark/Terminology: we say that we "select randomly" if after each selection/sample, the items remaining in the batch are equally likely to be selected.

Intersections, Multiplication Rules and Total Probability Rules

We know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

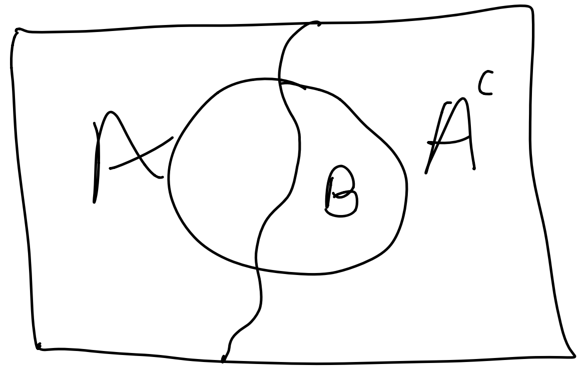
By rearranging the formula for conditional probability, we have a multiplication rule:

$$P(A \cap B) = P(B|A)P(A) = P(A|B)P(B).$$

Using the multiplication rules, we can sometimes determine the probability of an event by considering a partition of the sample space.

That is we sometimes find ourselves in the situation where we know the probability of an event, B, but only relative to other events.

Here is the picture:



The picture shows that

$$P(B) = P(A \cap B) + P(A' \cap B).$$

By the multiplication rule we can write

$$P(B) = P(B|A)P(A) + P(B|A')P(A').$$

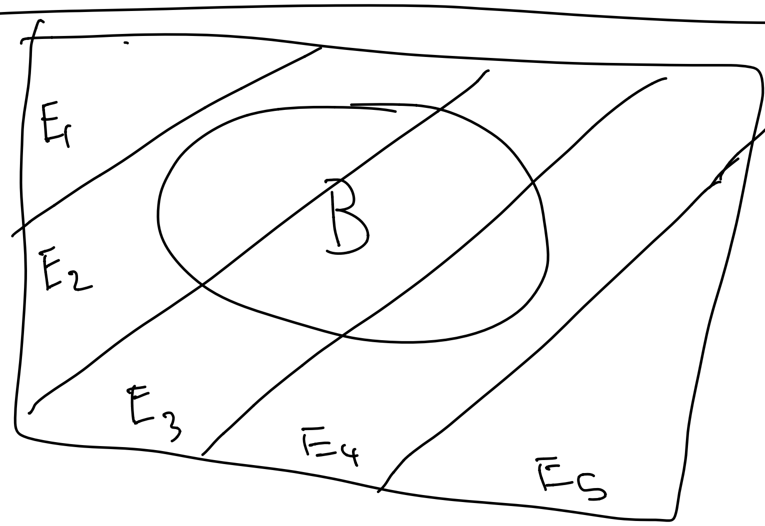
This is called the "law of total probability"

We have a more general version as well.

- Given a sample space S, we say that a set a collection of events $E_1, \ldots, E_n$ is exhaustive if $E_1 \cup \ldots \cup E_n = S$.
- If $E_1 \ldots E_n$ is an exhaustive list of mutually disjoint events, then for any event B we have a generalized law of total

probability:

$$P(B) = P(B\cap E_1) + P(B\cap E_2) + \dots + P(B\cap E_n)$$
$$= P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + \dots + P(B|E_1)P(E_1).$$



Ex: Suppose we shuffle a deck of cards.

Let B = " the second card is an ace".

We want to compute $P(B)$.

Let E_1 = " the first card is an ace"

$E_2 = E_1'$ = "The first card is not an ace".

By the law of total probabilities,

$$P(B) = P(B|E_1)P(E_1) + P(B|E_2)P(E_2).$$

So: $P(B|E_1) = \frac{3}{51}$. (since one ace is gone, and there are 3 left).

$P(E_1) = \frac{4}{52}$ (since there are 4 aces).

$P(B_1|E_2) = \frac{4}{51}$.

$P(E_2) = 48/52$

So:

$$P(B) = \left(\frac{3}{51}\right)\left(\frac{4}{52}\right) + \left(\frac{4}{51}\right)\left(\frac{48}{52}\right) = \frac{4}{52}.$$

Ex: Suppose we have 30 marbles split into three bags as follows:

Bag 1: 2 red, 4 green, 4 blue.

Bag 2: 4 red, 5 green, 1 blue.

Bag 3: 1 red, 2 green, 7 blue.

Assuming the bags are indistinguishable, what is the probability of choosing a red marble?

Let $R=$ "choose a red marble"

If the sample space is the set of marbles, then $B_i=$ "choose from the i^{th} bag"

gives an exhaustive, mutually exclusive partition of S: every marble is in at least one bag and no marble is in two bags.

So $P(R)=P(R|B_1)P(B_1)+P(R|B_2)P(B_2)+P(R|B_3)P(B_3)$.

Since the bags are indistinguishable,

$$P(B_1)=P(B_2)=P(B_3)=1/3.$$

So $P(R)=\left(\frac{2}{10}\right)\left(1/3\right)+\left(4/10\right)\left(1/3\right)+\left(1/10\right)\left(1/3\right)$

$$=\left(1/3\right)\left(\frac{2+4+1}{10}\right)=7/30.$$