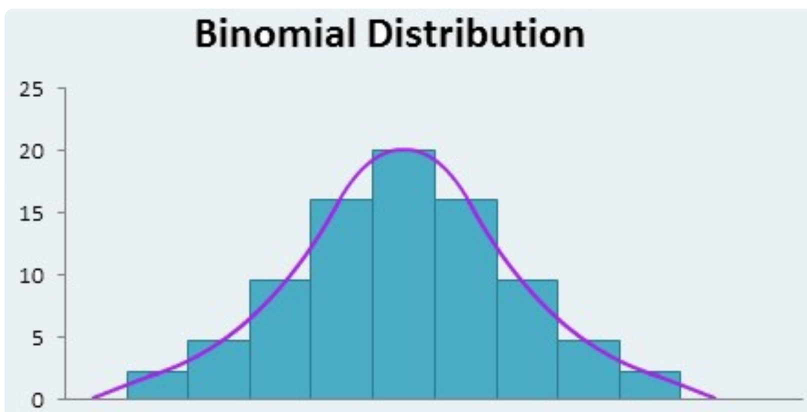
Lecture 8

Geometric and Negative Binomial Distributions

- Recall that given a series of Bernoulli trials (independent with fixed probability p of success per trial), the/a <u>binomial random variable</u>,

$$\text{Bin}(p, n) = \#\text{ of successful trials in } n\text{-trials.}$$



- The <u>Geometric distribution</u> is related.
- Rather than a fixed number of trials, we consider an arbitrary number of trials.
- A random variable X is called <u>geometric with parameter $0<p<1$</u> if

$$X = \text{"}\#\text{ of trials until the first successful trial"}.$$

- If X is a geometric random variable with parameter p, then X has a probability mass function given by

$$f(x) = (1-p)^{x-1} p. \quad x = 1, 2, 3, 4 \ldots$$

- Interpretation: $f(m)$ is the probability of $m-1$ failures in a row $(1-p)^{m-1}$ followed by a success (p).

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mean: If X is geometric with probability p

Then
$$E(X) = \sum_{x=1}^{\infty} x p (1-p)^{x-1}$$
$$= p \sum_{x=1}^{\infty} x (1-p)^{x-1}$$

Consider $\sum_{x=1}^{\infty} x(1-p)^{x-1}$. Substituting $q = 1-p$, we have

$$\sum_{x=1}^{\infty} x q^{x-1} = \frac{d}{dq}\left(\sum_{x=0}^{\infty} q^x\right)$$

geometric series, convergent since $q = 1-p < 1$

$$= \frac{d}{dq}\left(\frac{1}{1-q}\right)$$
$$= \frac{1}{(1-q)^2} = \frac{1}{p^2}.$$

so $\Rightarrow E(X) = p \cdot \left(\frac{1}{p^2}\right) = 1/p.$

Exercise/See text: $V(X) = \frac{1-p}{p^2}$.

Ex: Suppose that the probability that a bit transmitted through a digital transmission channel is received in error is 0.1.

- Assume the transmissions are independent events.

(See Example 3.17 in the text).

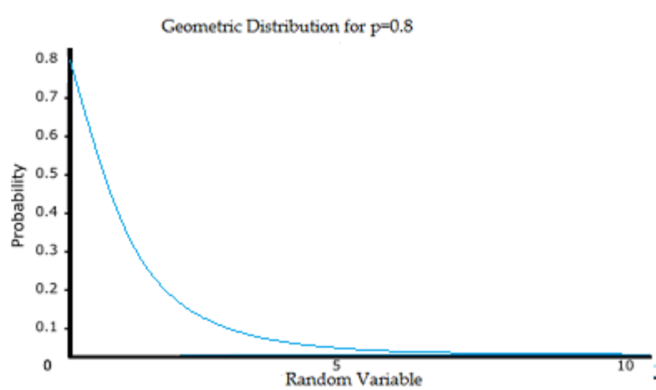
- Let X = "the number of bits transmitted until the first error".

- **Q:** What is $P(X \geq 11)$? i.e. what is the prob. that there are more that 10 bit transmitted without error / at least 10 bits without error?

note $\sum_{x=1}^{\infty} p(1-p)^{x-1} = 1$ geometric series.

$$P(X \geq 10) = 1 - P(X \leq 10)$$

$$= 1 - \sum_{x=1}^{10} (0.9)(0.1)^{x-1}$$



Negative Binomial Distribution

- A random variable X is geometric if X = "the number of bernoulli trials until a success."

- Negative binomial distributions generalize this.
  - Given some bernoulli trial, and some integer $r \geq 1$, we say that a DRV X is **negative binomial** if

    X = "the number of trials required to have r successful trials."

- Note that if X is negative binomial with $r=1$, then X is just geometric.

- Suppose that X is a negative binomial with parameters r and p. Then X has p.m.f. given by

$$f(x) = \binom{x-1}{r-1}(1-p)^{x-r}p^r$$

for $(x = r, r+1, r+2, \ldots)$.

- Let X be the number of trials required to get r successes.
- Let X_1 = "the number of trials required to get the first success" (out of r)
- Let X_i = " of trials to get the i^{th} success.
- Then $X = X_1 + X_2 + \dots + X_r$.
- It follows that:

$$\mu = E(X) = \frac{r}{p} \qquad \sigma^2 = V(X) = \frac{r(1-p)}{p^2}$$

Ex: "Inverse Binomial Sampling."

Suppose you, a red-head, are trying to form a chapter of RPFC, the Red-headed People's Front of Canada. Assume that you know no other red-heads, and so you decide to call random people until you find some red-heads to invite to your meeting.

Q: If no more than 2% of the population is red headed, what is the probability of talking to 10 red-heads if you make

50 calls? 500 calls? 1000 calls? n calls for n-large?

Soln: Let X be the random variable "# of people called until 10 red heads are called."

The PMF is

$$f(x) = \binom{x-1}{10-1}(1-0.02)^{x-10}(0.02)^{10}.$$

$f(50) \approx 9.3764 \times 10^{-9}$ (very tiny).

$f(500) \approx 0.0025$ (0.2%).

$f(1000) \approx 0.00005$ (small again!!)

$$\lim_{n\to\infty} f(n) = ?$$
$$= 0.$$

why? probability of getting exactly 10 red heads.

As n gets large there is high probability you will call more than 10.

In a random sample of 1000 people, you expect 20 red heads (2% of 1000).