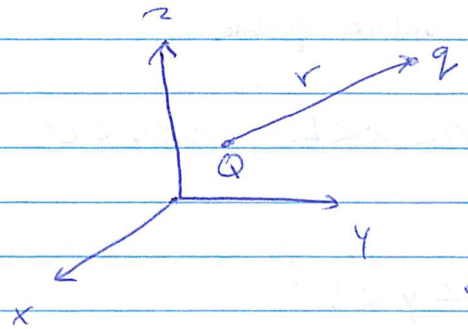
Electrostatic Potential.



Suppose we have two points with positive charges Q and q which are r units apart.

- Coulomb's law states that each repels the other with force whose magnitude is $F = \dfrac{qQ}{4\pi\varepsilon_0 r^2}$, where ε_0 = permittivity of free space (it's a constant)

- We can define the vector field (called electric field intensity) by

$$E = \frac{Q}{4\pi\varepsilon_0 r^3}\vec{r}$$

where $\vec{r}$ is the vector from Q to a unit charge q.

- We showed in examples from chapter 14 that $\nabla \times E = 0 = \nabla \cdot E$.
- $\nabla \times E = 0 \Rightarrow$ on a simply connected domain, $E = -\nabla V$ for some potential function V.

- Since $\nabla \cdot E = 0 \Rightarrow \nabla \cdot (-\nabla V) = -\nabla^2 V = 0 \Rightarrow \nabla^2 V = 0$ which is Laplace's equation.

- For this specific E, we can simply solve $E = -\nabla V$ to find V.

For more general charge distributions, Laplace's equation still holds outside the distribution, and Poisson's equation ($\nabla^2 V = -\frac{6}{\varepsilon}$) holds inside. More details are provided in section 20.3

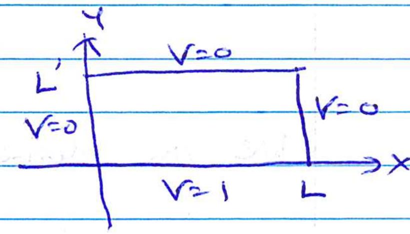
- You have seen Laplace's equation in Math 2130. Solutions to this differential equation were said to be harmonic.

Example: Solve the boundary value problem

$$\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \quad 0 < x < L, \quad 0 < y < L',$$

$$V(0,y) = 0 = V(L,y) \qquad 0 < y < L'$$
$$V(x,L') = 0, \; V(x,0) = 1 \qquad 0 < x < L$$

for potential in the rectangular plate when sides are maintained at the above potentials.



Using separation of variables, write $V(x,y) = X(x)Y(y)$ so that

$$\frac{\partial^2 X}{\partial x^2} Y + \frac{\partial^2 Y}{\partial y^2} X = 0$$

$$\Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y}$$

$$\Rightarrow \quad X'' + \lambda X = 0 \quad \text{and} \quad Y'' - \lambda Y = 0$$ using $-\lambda$ in the separation principle.

The boundary conditions imply $X(0) = 0 = X(L)$ and $Y(L') = 0$.

Using information from our computations in chapter 19, we see that $\lambda_n = \frac{n^2 \pi^2}{L^2}$, $n \geq 1$ are eigenvalues for

$X'' + \lambda X = 0$ and a solution has the form

$$C \sin\left(\frac{n\pi x}{L}\right)$$ for some integer $n \geq 1$.

The second system $Y'' - \lambda Y = 0$ has auxiliary equation $m^2 - \lambda = 0$, but λ from the previous computation is $\frac{n^2\pi^2}{L^2}$, so $m = \pm \frac{n\pi}{L}$

Using our knowledge of ODE solutions, we see that a general solution is

$$Y(y) = De^{\frac{n\pi y}{L}} + Fe^{-\frac{n\pi y}{L}}.$$

The boundary condition $\Rightarrow$ $E = -De^{\frac{2n\pi L'}{L}}$

$$\Rightarrow Y(y) = D\left[e^{n\pi y/L} - e^{n\pi(2L'-y)/L}\right]$$

$$\Rightarrow V(x,y) = X(x)Y(y)$$
$$= b\sin\left(\frac{n\pi x}{L}\right)\left[e^{n\pi y/L} - e^{n\pi(2L'-y)/L}\right]$$

with $b \in \mathbb{R}$. This is a solution to Laplace's equation, and since all of the conditions are homogeneous, we can use superposition to get

$$V(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\left[e^{n\pi y/L} - e^{n\pi(2L'-y)/L}\right]$$

The final condition $V(x,0) = 1$ implies

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) = L, \qquad 0 < x < L$$

with $C_n = b_n\left(1 - e^{2n\pi L'/L}\right)$.

- We can recognize this as the eigenfunction expansion of the function 1.

Then $C_n = \frac{2}{L}\int_0^L (1)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2\left[1+(-1)^{n+1}\right]}{n\pi}$

$$\Rightarrow b_n = \frac{2\left[1+(-1)^{n+1}\right]}{\left(1 - e^{2n\pi L'/L}\right)}$$

This series can be simplified (?) to the one on line (21.26).

Try working through Part C in 21.2 for practice with potential problems.