Sample Exam B

1. Evaluate the line integral

$$\int_C \left(3x^2yz + \frac{1}{x+y}\right) dx + \left(x^3z + \frac{1}{x+y}\right) dy + (x^3y + 2z)\, dz$$

where C is the straight line from $(1,1,1)$ to $(3,4,6)$.

2. Evaluate the line integral

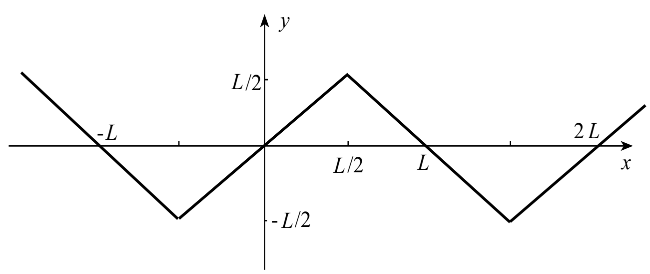
$$\oint_C [(2xe^y + y^3)\hat{\mathbf{i}} + (x^2e^y + z)\hat{\mathbf{j}} + xz^2\,\hat{\mathbf{k}}] \cdot \mathbf{dr}$$

once around the curve

$$C: \quad z = x^2 + y^2, \quad z = 10 - 4x^2 - 4y^2$$

directed counterclockwise as viewed from the origin.

3. Find the Fourier series of the function in the figure below. Simplify the series as much as possible.



4. (a) Expand the function $f(x) = 3x$, $0 < x < 2$ in terms of the eigenfunctions of the Sturm-Liouville system

$$\frac{d^2y}{dx^2} + \lambda y = 0, \quad 0 < x < 2,$$
$$y(0) = 0,$$
$$y'(2) = 0.$$

(b) Does the series in part (a) converge to $f(x)$ at $x = 0$ and at $x = 2$? You may use the fact that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

5. (a) Find all singular points for the differential equation

$$(x^2 - 4)y'' + (x+2)\,y' + 3y = 0.$$

Are they regular singular, or irregular singular? Justify your answers.

(b) If a Taylor series $\sum_{n=0}^{\infty} a_n (x-1)^n$ is found for the solution of the differential equation in part (a), what would you predict for its radius of convergence. Justify your answer.

6. (a) Show that the indicial roots for the Frobenius solution $\sum_{n=0}^{\infty} a_n x^{n+r}$ of the differential equation

$$xy'' + 2y' - 3y = 0$$

differ by an integer.

(b) Find the solution of the differential equation corresponding to the smaller indicial root. Express your answer in sigma notation simplified as much as possible. Is it a general solution? What is the radius of convergence of the series?

7. Solve the following initial, boundary-value problem

$$\begin{aligned} \frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \\ U_x(0,t) &= 0, \quad t > 0, \\ U(L,t) &= 0, \quad t > 0, \\ U(x,0) &= x, \quad 0 < x < L. \end{aligned}$$

Justify each step in your solution.

8. (a) The initial, boundary-value problem

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} &= c^2\frac{\partial^2 y}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \\ y_x(0,t) &= k, \quad t > 0, \\ y(L,t) &= 0, \quad t > 0, \\ y(x,0) &= f(x), \quad 0 < x < L, \\ y_t(x,0) &= g(x), \quad 0 < x < L, \end{aligned}$$

describes displacements of a taut string. Constant k arises from a force acting on the left end of the string which moves vertically. Because the problem has nonhomogeneities, it is necessary to split $y(x,t)$ into two parts $y(x,t) = z(x,t) + \psi(x)$, where $\psi(x)$ represents the static deflection of the string. Find $\psi(x)$.

(b) Find the initial, boundary-value problem satisfied by $z(x,t)$. Do **NOT** attempt to solve this problem.

Answers:

1. $682 + \ln(7/2)$ **2.** 3π **3.** $\dfrac{4L}{\pi^2}\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{(2n-1)^2}\sin\dfrac{(2n-1)\pi x}{L}$

4.(a) $\dfrac{48}{\pi^2}\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{(2n-1)^2}\sin\dfrac{2n-1)\pi x}{4}$, (b) Yes

5.(a) $x = \pm 2$ are regular singular points (b) 1

6.(b) $y(x) = a_1\sum_{n=0}^{\infty}\dfrac{3^n}{n!(n+1)!}x^n$ Not general ∞

7. $U(x,t) = \sum_{n=1}^{\infty}\left[\dfrac{4L(-1)^{n+1}}{(2n-1)\pi} - \dfrac{8L}{(2n-1)^2\pi^2}\right]e^{-(2n-1)^2\pi^2 kt/(4L^2)}\cos\dfrac{(2n-1)\pi x}{2L}$

8.(a) $\psi(x) = k(x-L)$

(b)

$$\begin{aligned}
\frac{\partial^2 z}{\partial t^2} &= c^2\frac{\partial^2 z}{\partial x^2}, \quad 0 < x < L, \quad t > 0,\\
z_x(0,t) &= k, \quad t > 0,\\
z(L,t) &= 0, \quad t > 0,\\
z(x,0) &= f(x) - k(x-L), \quad 0 < x < L,\\
z_t(x,0) &= g(x), \quad 0 < x < L,
\end{aligned}$$