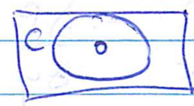
Example 5: If D is simply connected, and $\nabla \times F = 0$ in D, show there exists a function $f(x,y,z)$ such that $\nabla f = F$ in D (the sufficient condition in Theorem 14.1)

If C is any piecewise-smooth closed curve in D, then there exists a piecewise-smooth surface S in D with C as the boundary (this is using the fact that D is simply connected - you could not find such an S for  xy-plane).

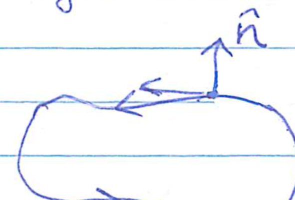
By Stokes's theorem

$$\oint_C F \cdot dr = \iint_S (\nabla \times F) \cdot \hat{n} \, dS = 0$$

This is true for any closed curve C in D $\Rightarrow \int F \cdot dr$ is independent of path.

$\Rightarrow \exists f$ where $\nabla f = F$ in D by Theorem 14.3

Example 6: Rewrite Green's theorem using the divergence theorem.



Given a curve $x = x(t)$, $y = y(t)$, a tangent vector can be found using $(x'(t), y'(t))$ and a normal outer vector is $(y'(t), x'(t)) = n$.

Then $$\hat{n} = \frac{(y'(t), x'(t))}{\sqrt{(x'(t))^2 + (y'(t))^2}}$$

Let $F(x,y) = Q(x,y)\,\hat{i} + (-P(x,y))\,\hat{j}$. Then

$$F \cdot \hat{n} \, ds = \left[\frac{Q(x,y)\, y'(t) + P(x,y)\, x'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}}\right] \sqrt{(x'(t))^2 + (y'(t))^2} \, dt$$

$$= Q(x,y)\,y'(t)\,dt + P(x,y)\,x'(t)\,dt$$
$$= P(x,y)\,dx + Q(x,y)\,dy.$$

Then $\oint_C P\,dx + Q\,dy = \oint_C F\cdot\hat{n}\,ds$

On the other hand,

$$\nabla\cdot F = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

so $\iint_R \nabla\cdot F\,dA = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$

so Green's theorem can be written as

$$\oint_C F\cdot\hat{n}\,ds = \iint_R \nabla\cdot F\,dA.$$

The divergence theorem in general can be stated as

$$\underbrace{\int\cdots\int_M}_{n} \nabla\cdot F\,dV = \underbrace{\oint\cdots\oint_{\partial M}}_{n-1} F\cdot\hat{n}\,dS,$$

M is n-dimensional and ∂M is $n-1$ dimensional.

or $\int_M \nabla\cdot F\,dV = \oint_{\partial M} F\cdot\hat{n}\,dS$

- If $n=2$, we get Green's theorem
- If $n=1$, then take $M=[a,b]$, $\partial M=\{a,b\}$, "0-dim"
  $F(x)=f(x)\hat{i}$ so $\nabla\cdot F=\frac{\partial f}{\partial x}$ and RHS is a V integral over each point

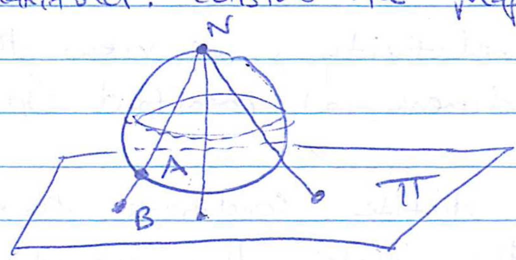
$$\int_a^b \frac{\partial f}{\partial x}\,dx = f(b) - f(a)$$

- In this way, we can see that the generalization of the FTC is Green's theorem for $n=2$, and the divergence theorem or Stokes's theorem as two generalizations, one using the divergence, the other using the curl.

Note: The remainder of this section will not be tested. It is meant as an introduction to more general language.

An n-dimensional manifold, roughly defined, is a space that can be locally described as $\mathbb{R}^n$. That is, in an open set around each point, M behaves like $\mathbb{R}^n$.

For example, the sphere $S: x^2+y^2+z^2=1$ is a 2-dimensional manifold. Consider the map which takes a line from the north pole to the xy-plane, and uses it to map A on S to B on Π. This is the stereographic projection.



- In this way, there is a bijection between $S \setminus \{N\}$ and $\mathbb{R}^2$. We could do the same for the south pole, to cover S with two "coordinate charts."

- All the surfaces we have encountered (including the Mobius strip) are manifolds.

- If you do the same thing except with $\mathbb{R}^n_{\{x_n \geq 0\}}$ (half-space), then you get a "Manifold with boundary".

$B: x^2+y^2+z^2 \leq 1$ is a 3-dimensional manifold with boundary $\partial B = S$.

- Using this language, we have already generalized the divergence theorem.

- Stokes's theorem can be written as,

$$\int_{\partial M} \omega = \int_M d\omega$$ where ω is a differential form on M (think of vector fields for a base example).

- we showed for surfaces that

$$dS = |\sigma_u \times \sigma_v| \, du \, dv$$

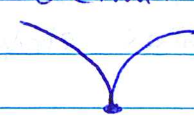
tangent vectors.

Generalizations of this involve "tangent bundles" and "volume forms".

- Ultimate goal: Can we do calculus on a manifold?

For example, in general relativity, we view the universe as a 4-dimensional manifold (space-time).

Einstein field equations dictate conditions that this manifold must satisfy (they relate the geometry of M with the distribution of mass within it)

In this way, we have numerous possibilities for M. In certain cases, singularities can occur in the definition (slightly different from the manifold definition), singularities in the geometry just like  we have seen in past classes.

These are what we call black-holes.

17.1 Series Solutions of Differential Equations

A function $y = f(x)$ is said to be analytic on some interval I (called the interval of convergence) if it can be locally defined by a convergent power series.

That is, $\forall$ for any $x_0 \in I$, $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ for $x \in I$.

This is equivalent to being an infinitely differentiable function whose Taylor series at $x_0 \in I$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$ converges (pointwise) to $f(x)$ for x near x_0

In other words the sequence $S_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \longrightarrow f(x)$

- We denote the set of analytic functions by $C^{\omega}(I)$ and ∞-differentiable functions by $C^{\infty}(I)$.

- $C^{\omega}(I) \subset C^{\infty}(I)$ but are not equal over the real numbers. (For example $g(x) = \begin{cases} e^{-1/x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is in C^{∞} but not C^{ω})

- All elementary functions are analytic on some interval.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad -\infty < x < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \qquad -\infty < x < \infty$$

$$\frac{a}{1-x} = a + ax + ax^2 + ax^3 + \dots \qquad -1 < x < 1$$

- On I, we can differentiate and integrate term by term

- These power series allow us to approximate functions by polynomials. Let us apply this to ODEs.

Example 1: Find a solution to the differential equation

$$\frac{dy}{dx} - 2y = 0.$$

Method 1: The equation is separable, so

$$\frac{dy}{dx} = 2y \quad \Rightarrow \quad \frac{dy}{2y} = dx \qquad \text{for } y \neq 0$$

$$\Rightarrow \int \frac{dy}{2y} = \int dx$$

$$\Rightarrow \frac{\ln|2y|}{2} = x + C_0$$

$$\Rightarrow \ln|2y| = 2x + C_1 \quad , \quad C_1 = 2C_0$$

$$|2y| = e^{2x+C_1}$$

$$y = \frac{\pm e^{C_1}}{2} e^{2x} = Ce^{2x}$$

where $C \in \mathbb{R}$ is some constant

($C=0$ corresponds to $y=0$ solution)

Method 2: Assume that the solution to $\frac{dy}{dx} - 2y = 0$ is analytic on some interval.

Then $y(x) = \sum_{n=0}^{\infty} a_n x^n$, and

$$\frac{dy(x)}{dx} = \frac{d}{dx}(a_0) + \frac{d}{dx}(a_1 x) + \frac{d}{dx}(a_2 x^2) + \cdots$$

on the interval of convergence

$$= a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

So $\frac{dy}{dx} - 2y = (a_1 + 2a_2 x + 3a_3 x^2 + \cdots) + (-2a_0 - 2a_1 x - 2a_2 x^2 - \cdots$

$$\Rightarrow \quad 0 = (a_1 - 2a_0) + (2a_2 - 2a_1)x + (3a_3 - 2a_2)x^2 + \cdots$$

$$\Rightarrow \quad \begin{aligned} a_1 - 2a_0 &= 0 \\ 2a_2 - 2a_1 &= 0 \\ 3a_3 - 2a_2 &= 0 \\ 4a_4 - 2a_3 &= 0 \\ &\vdots \end{aligned} \quad \Rightarrow \quad \begin{aligned} a_1 &= 2a_0 \\ a_2 &= a_1 = 2a_0 \\ a_3 &= \frac{2}{3}a_2 = \frac{2^2 a_0}{3} \\ a_4 &= \frac{2}{4}a_3 = \frac{2^3 a_0}{4 \cdot 3} \\ &\vdots \end{aligned}$$

We can rewrite $\dfrac{2^3}{4 \cdot 3} = \dfrac{2^4}{4 \cdot 3 \cdot 2} = \dfrac{2^4}{4!}$

so in general, $a_n = \dfrac{2^n a_0}{n!}$ (we can prove this by induction)

Then
$$y(x) = \sum_{n=0}^{\infty} \frac{2^n a_0 x^n}{n!} = a_0 \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$
$$= a_0 \sum_{n=0}^{\infty} \frac{y^n}{n!} \quad \text{if } y = 2x$$
$$= a_0 e^y = a_0 e^{2x}$$

Where $a_0 \in \mathbb{R}$ is some constant.

Example 2: Find a solution to the differential equation
$$y'' + xy' - y = 0.$$

- Assume the solution has a Maclaurin Series $y(x) = \sum_{n=0}^{\infty} a_n x^n$ with ROC, R

Then
$$0 = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^{\overset{n-1+1}{n}} + \sum_{n=0}^{\infty} -a_n x^n$$

- By Theorem 10.3, each series has radius of convergence (ROC) of R.

The $n=k$ term for the first series has exponent $k-2$, for the second series k and same for the third series.