$$\Rightarrow \begin{aligned} a_1 - 2a_0 &= 0 \\ 2a_2 - 2a_1 &= 0 \\ 3a_3 - 2a_2 &= 0 \\ 4a_4 - 2a_3 &= 0 \\ &\vdots \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} a_1 &= 2a_0 \\ a_2 &= a_1 = 2a_0 \\ a_3 &= \frac{2}{3}a_2 = \frac{2^2 a_0}{3} \\ a_4 &= \frac{2}{4}a_3 = \frac{2^3 a_0}{4\cdot 3} \\ &\vdots \end{aligned}$$

We can rewrite $\dfrac{2^3}{4\cdot 3} = \dfrac{2^4}{4\cdot 3\cdot 2} = \dfrac{2^4}{4!}$

so in general, $a_n = \dfrac{2^n a_0}{n!}$ (we can prove this by induction)

Then
$$y(x) = \sum_{n=0}^{\infty} \frac{2^n a_0 x^n}{n!} = a_0 \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$
$$= a_0 \sum_{n=0}^{\infty} \frac{y^n}{n!} \quad \text{if } y = 2x$$
$$= a_0 e^y = a_0 e^{2x}$$

Where $a_0 \in \mathbb{R}$ is some constant.

Example 2: Find a solution to the differential equation
$$y'' + xy' - y = 0.$$

- Assume the solution has a Maclaurin Series $y(x) = \sum_{n=0}^{\infty} a_n x^n$ with ROC, R

Then
$$0 = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^{\overset{n-1+1}{n}} + \sum_{n=0}^{\infty} -a_n x^n$$

- By Theorem 10.3, each series has radius of convergence (ROC) of R.

The $n=k$ term for the first series has exponent $k-2$, for the second series k and same for the third series.

Shift the first series using $n = j+2$
so that $n=0 \Rightarrow j=-2$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{j=-2}^{\infty} (j+2)(j+1)a_{j+2}x^j$$
$$= \sum_{j=0}^{\infty} (j+2)(j+1)a_{j+2}x^j$$

since the first 2 terms are 0

Then
$$\sum_{j=0}^{\infty} (j+2)(j+1)a_{j+2}x^j + \sum_{n=0}^{\infty} na_n x^n + \sum_{n=0}^{\infty} -a_n x^n$$
$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + na_n - a_n\right]x^n$$

which also has a radius of convergence of R.

- For this to be 0, we must have each

$$(n+2)(n+1)a_{n+2} + (n-1)a_n = 0, \quad n \geq 0$$

$$\Rightarrow \quad a_{n+2} = \frac{(1-n)a_n}{(n+2)(n+1)}$$

$n=0 \qquad a_2 = \frac{a_0}{2}$

$n=1 \qquad a_3 = 0$

$n=2 \qquad a_4 = \frac{-a_2}{4\cdot 3} = \frac{-a_0}{4\cdot 3\cdot 2} = \frac{-a_0}{4!}$

$n=3 \qquad a_5 = \frac{-2a_3}{5\cdot 4} = 0$

$n=4 \qquad a_6 = \frac{-3a_4}{6\cdot 5} = \frac{+3a_0}{6\cdot 5\cdot 4!} = \frac{3a_0}{6!}$

$n=5 \qquad a_7 = 0$

$n=6 \qquad a_8 = \frac{-5a_6}{8\cdot 7} = \frac{-5\cdot 3a_0}{8\cdot 7\cdot 6!} = \frac{-5\cdot 3a_0}{8!}$

What we know so far is

$$y(x) = a_0 + a_1 x + \frac{a_0}{2!}x^2 - \frac{a_0}{4!}x^4 + \frac{3a_0}{6!}x^6 - \frac{3\cdot 5 a_0}{8!}x^8 + \cdots$$

$$= a_1 x + a_0\left[1 + \frac{x^2}{2!} + \sum_{n=2}^{\infty}(-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{(2n)!}x^{2n}\right]$$

but $\frac{1\cdot 3\cdot 5\cdots(2n-3)}{(2n)!} = \frac{1\cdot 2\cdot 3\cdot 4\cdot\ldots\cdot(2n-3)}{(2n)!\,2\cdot 4\cdot 6\cdot\ldots\cdot(2n-4)}$

$$= \frac{(2n-3)!}{(2n)!\,2^{n-2}\cdot 1\cdot 2\cdot 3\cdot\ldots\cdot(n-2)}$$

$$= \frac{(2n-3)!}{2^{n-2}(2n)!(n-2)!}$$

$$= \frac{(2n-3)!}{2^{n-2}(2n)(2n-1)(2n-2)(2n-3)!(n-2)!}$$

$$= \frac{1}{2^{n-2}\cdot 2^2\cdot n(2n-1)(n-1)(n-2)!}$$

$$= \frac{1}{2^n(2n-1)n!}$$

Then $y(x) = a_1 x + a_0 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^n(2n-1)n!} x^{2n}$ two linearly independent solutions

(Notice that for $n=0$ we have $\frac{(-1)a_0}{1(-1)1} = a_0$ ✓

and $n=1$ $\frac{a_0(-1)^2 x^2}{2(2-1)1} = \frac{a_0 x^2}{2}$)

What's the interval of convergence for this second series?

By the ratio test, since $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{\left|\frac{(-1)^{n+1}}{2^n(2n-1)n!}\right|}{\left|\frac{(-1)^{n+2}}{2^{n+1}(2n+1)(n+1)!}\right|}$

quadratic

$= \lim_{n\to\infty}\frac{2\cdot(2n+1)(n+1)}{2n-1} = \infty$

linear

then $R = \infty \Rightarrow y(x)$ is general solution

Review of Linear Differential Equations

An n^{th} order linear differential equation has the form (defined on some interval I),

$$F(x) = a_0(x)\,y^{(n)}(x) + a_1(x)\,y^{(n-1)}(x) + \dots + a_{n-1}(x)\,y'(x) + a_n(x)\,y$$

with $a_0(x) \neq 0$. It is homogeneous if $F(x) = 0$ and nonhomogeneous otherwise.

- It is linear because $(a_1(x)D^n + \dots + a_n(x))$ is an operator applied to y, and is linear in the sense of linear algebra.
- The superposition principle states that if $y_1(x)$ and $y_2(x)$ are solutions, then so is $c_1y_1 + c_2y_2$ (homogeneous case)
- It turns out that the set of solutions of a homogeneous linear differential equation of order n forms a vector space of dimension n. (over $\mathbb{R}$)
  (I has to be restricted if $a_0(x) = 0$ at some points).
  $\Rightarrow$ exists n linearly independent solutions which span the vector space. ie, we can find a basis $y_1(x), \dots, y_n(x)$ so that
  $y(x) = c_1y_1(x) + \dots + c_ny_n(x)$ is a general solution
  $c_1, \dots, c_n \in \mathbb{R}$.
- We can classify solutions for the homogeneous case with constant coefficients (section 15.8). This can be used to find a general solution in the constant coefficient nonhomogeneous case (section 15.9)
- For the non-constant coefficient case, solutions can be more difficult to find. This is where power series techniques work well.

General procedure: Assume there is some analytic solution. Use this to find a power series which satisfies the differential equation. Check that it has a positive radius of convergence. Then the function is a solution (vindicated by the previous computations) although it may not be a general solution.

Example 3: Find all analytic solutions at the origin to $x^3y''+y=0$.

Assume there is a solution centered at 0, $y(x)=\sum_{n=0}^{\infty} a_n x^n$.

Then
$$0 = \sum_{n=0}^{\infty} n(n-1)a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^n$$
$$= \sum_{i=1}^{\infty} (i-1)(i-2)a_{i-1}x^i + \sum_{n=0}^{\infty} a_n x^n \qquad i=n+1$$

Shift so the powers of x are determined by the index without shift.

$$= a_0 + \sum_{n=1}\left[(n-1)(n-2)a_{n-1}+a_n\right]x^n$$

$$\Rightarrow a_0 = 0 \quad \text{and} \quad (n-1)(n-2)a_{n-1}+a_n = 0 \quad n\geq 1$$

$$\Rightarrow a_n = -(n-1)(n-2)a_{n-1} \quad \text{so} \quad a_1=0$$
$$a_2=0$$
$$a_3=0 \quad \text{etc.}$$

In this case $y(x)\equiv 0$. identically 0

This means that any solution to the ODE will not be analytic at the origin (for example $e^{-1/x}$ mentioned before)

Example 4: Find all solutions to $4xy''+(2+2x)y'+y=0$ which are analytic at the origin.

Let $y(x)=\sum_{n=0}^{\infty} a_n x^n$ be a supposed solution.

Then $0 = \sum_{n=0}^{\infty} 4n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} 2na_n x^{n-1} + \sum_{n=0}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} a_n x^n$

$$= \sum_{n=-1}^{\infty} 4(n+1)n\,a_{n+1} x^n + \sum_{n=-1}^{\infty} 2(n+1)a_{n+1} x^n + \sum_{n=0}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} \left[4(n+1)n\,a_{n+1} + 2(n+1)a_{n+1} + 2na_n + a_n\right] x^n$$

$$= \sum_{n=0}^{\infty} \left[2(n+1)(2n+1)a_{n+1} + (2n+1)a_n\right] x^n$$

Equating to 0 gives $a_{n+1} = \dfrac{-a_n}{2(n+1)}, \quad n \geq 0$

When $n=0$, $a_1 = \dfrac{-a_0}{2}$.

$n=1$, $a_2 = \dfrac{-a_1}{2(2)} = \dfrac{+a_0}{2^2(2)}$

$n=2$, $a_3 = \dfrac{-a_2}{2(3)} = \dfrac{-a_0}{2^3(3!)}$

It looks like $a_n = \dfrac{(-1)^n a_0}{2^n n!}$.

We could check a few more terms to be sure.
We could also show it using mathematical induction.

Base case, $n=0$, $a_0 = \dfrac{(-1)^0 a_0}{2^0 0!} = a_0$ ✓ ($0! = 1$ by the way)

Assume it's true for $n=k$. Then for $n=k+1$ we have

$$\underbrace{a_{k+1} = \frac{-a_k}{2(k+1)}}_{\text{what we know is true from the series}} = \frac{-1}{2(k+1)}\left(\frac{(-1)^k a_0}{2^k k!}\right)$$

by assumption

$$= \frac{(-1)^{k+1} a_0}{2^{k+1}(k+1)!}$$

So the formula is true by induction.